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Orbifolds and groupoids

Matteo Tommasini [1]

Mathematical Physics Sector – SISSA, Via Bonomea 265, 34136 Trieste, Italy

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ABSTRACT

We define a 2-category structure (**Pre-Orb**) on the category of reduced complex orbifold atlases. We construct a 2-functor F from (**Pre-Orb**) to the 2-category (**Grp**) of proper étale effective groupoid objects over the complex manifolds. Both on (**Pre-Orb**) and (**Grp**) there are natural equivalence relations on objects: (a natural extension of) equivalence of orbifold atlases on (**Pre-Orb**) and Morita equivalence in (**Grp**). We prove that F induces a bijection between the equivalence classes of its source and target.

0. Introduction

A well-known issue in mathematics is that of modeling geometric objects where points have non-trivial groups of automorphisms. When the groups associated to every point are finite, the standard approach in topology and differential/complex geometry is through orbifolds. In particular, we need orbifolds in order to study quotients of manifolds by the action of finite groups of automorphisms; analogous to the definition of manifold, a complex orbifold is locally modeled on an open subset of $\mathbb{C}^n$ modulo a finite group of biholomorphisms acting on it. There is a well-defined notion of "map" between orbifolds (for example one can adapt the definition of the appendix of [5] from the real to the complex case) and composition of them, so orbifolds form a category.

On the other hand, in algebraic complex geometry, objects that have non-trivial group of automorphisms arise frequently from moduli problems and are usually studied as (Deligne–Mumford) algebraic stacks, that form in a natural way a non-trivial 2-category. A third approach, intermediate between the previous two, is the one that uses smooth groupoid objects, which also form a 2-category; for an introduction to these objects, see for example [11]. There exist strong relations between groupoid objects over the category of schemes and algebraic stacks, as shown in [8] (for a short introduction to these objects, look also at the appendix of [20]); analogous strong relations were found by D. Pronk in [16] and [17] for the topological and differentiable case.

Moreover, there is a very good reason to think of orbifolds as groupoid objects (at least in the smooth case) because of a construction due to D. Pronk [16] that allows us to associate to every smooth reduced orbifold atlas a proper étale

E-mail address: matteo.tommasini2@gmail.com.

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and effective groupoid object over smooth manifolds. Hence it seems natural to try to give a richer structure to complex orbifolds, i.e. to make them into a non-trivial 2-category and to see if there exists any relation between morphisms and 2-morphisms for orbifolds and the corresponding ones for groupoid objects over complex manifolds.

For simplicity in this work we will always restrict our attention to the complex case and to effective actions, i.e. all the orbifold atlases will be reduced and all the groupoid objects will be effective. This article is divided in four sections as follows:

(1) We review the basic facts about 2-categories and objects and morphisms in the category of reduced complex orbifold atlases; then we prove that this category can be embedded into a non-trivial 2-category, called (**Pre-Orb**) by defining suitable 2-morphisms (that will be called natural transformations), 2-identities and vertical and horizontal compositions of them. In addition, we review the definition of equivalence of orbifold atlases, analogous to the notion of compatibility between manifold atlases.
(2) A straightforward calculation proves that there exists a natural 2-category (**Grp**) where the objects are proper étale and effective groupoid objects over complex manifolds and morphisms and 2-morphisms are the usual morphisms and 2-morphisms between any pair of groupoid objects.
(3) The construction due to D. Pronk can be adapted from the smooth to the complex case in order to associate to every object of (**Pre-Orb**) an object of (**Grp**). Moreover, we prove that this construction can be extended to morphisms and 2-morphisms in order to get a 2-functor $F : (\textbf{Pre-Orb}) \to (\textbf{Grp})$.
(4) We review briefly the notion of Morita equivalence on groupoid objects. Then we prove that F induces a *bijection* between classes of orbifold atlases (as described in Section 1) and classes of Morita equivalent groupoid objects. Moreover, it is implicitly proved that if one considers the 2-functor U (described in [17]) of localization (up to Morita equivalence) of (**Grp**), we get that $U \circ F$ is essentially surjective.

Note that we don't call the first 2-category (**Orb**) or (**Orbifolds**) because its objects would have been orbifolds (i.e. equivalence classes of orbifold atlases, see Section 1.6). Indeed, there remain the following 2 open problems:

(a) What is the natural extension of the equivalence in Section 1.6 to morphisms and 2-morphisms of orbifolds in order to describe a 2-category (or a bicategory) (**Orb**)? Is it possible to use the calculus of fractions of D. Pronk in order to formally invert the (functorial) refinements of atlases?
(b) Is this extension compatible with F? In other words, consider the bicategory $(\textbf{Grp})[W^{-1}]$ obtained by inverting the class W of Morita equivalences using the calculus of fractions. Then suppose that (a) is solved; is it possible to use the results of Section 4 in order to induce a 2-functor $\tilde{F}$ from (**Orb**) to $(\textbf{Grp})[W^{-1}]$?

1. The 2-category of complex reduced orbifold atlases

1.1. 2-categories and 2-functors

We assume the standard notions of categories, (covariant) functors, fiber products in a fixed category and natural transformations (see, for example [2]). In this work we will also use the notions of 2-categories and 2-functors, that we recall briefly:

Definition 1.1. ([2], Def. 7.1.1) A 2-*category* $\mathscr{A}$ consists of the following data:

(1) A *class* $\mathscr{A}_0$, whose elements are called objects.
(2) For every pair of objects A, B, a *small category* $\mathscr{A}(A, B)$; the objects of this category are called morphisms or 1-morphisms and will be denoted by $f : A \to B$. The morphisms of this category between any pair of 1-morphisms f and g are called 2-morphisms and are denoted by $\alpha : f \Rightarrow g$. The composition of 2 composable morphisms α, β in the category $\mathscr{A}(A, B)$ is called *vertical* composition and denoted with $\beta \odot \alpha$.
(3) For each triple A, B, C of objects of $\mathscr{A}$, a functor *composition* from $\mathscr{A}(A, B) \times \mathscr{A}(B, C)$ to $\mathscr{A}(A, C)$. The composition of two objects (f, g) of the product is denoted by $g \circ f$; the composition of two morphisms (α, β) is called *horizontal* composition and denoted by $\beta * \alpha$.
(4) For each object A of $\mathscr{A}$, a morphism 1_A on A and a 2-morphism i_A on 1_A.

We require that these data satisfy the following axioms (which are not the original axioms of [2], but are equivalent to them):

(a) for every triple of 1-morphisms of the form $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ we have $(h \circ g) \circ f = h \circ (g \circ f)$;
(b) for every triple of horizontally composable 2-morphisms α, β, γ we have $(\gamma * \beta) * \alpha = \gamma * (\beta * \alpha)$;
(c) for each 1-morphism $A \xrightarrow{f} B$, we have $f \circ 1_A = f = 1_B \circ f$;
(d) for each 2-morphism $\alpha : (f : A \to B) \Rightarrow (g : A \to B)$ we require that $\alpha * i_A = \alpha = i_B * \alpha$.

For some basic examples of 2-categories, see [2], Example 7.1.4 (the most simple example is the 2-category of small categories, functors and natural transformations between them).

Definition 1.2. (Equivalent to [2], Def. 7.2.1.) Given two 2-categories $\mathscr{A}$ and $\mathscr{B}$, a (*covariant*) 2-*functor* $F : \mathscr{A} \to \mathscr{B}$ consists of the following data:

(1) For each object A in $\mathscr{A}$, an object $F(A)$ in $\mathscr{B}$.
(2) For each pair of objects A, A', a functor $F_{A,A'} : \mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A'))$; with a little abuse of notation, sometimes we will denote this functor only with F. These data must satisfy the following axioms:
 (a) F preserves compositions of 1-morphisms;
 (b) F preserves compositions of 2-morphisms;
 (c) for every object A of $\mathscr{A}$, $F(1_A) = 1_{F(A)}$ and $F(i_A) = i_{F(A)}$.

1.2. Uniformizing systems, embeddings and atlases

Let us review some basic definitions about complex orbifolds. All the definitions of this section are just translations to the complex case of the corresponding definitions for the smooth case (see, for example, the appendix of [5]). Since we will work only in the holomorphic case, in general we will use the word “orbifold” instead of “complex orbifold”.

Definition 1.3. Let X be a paracompact second countable Hausdorff topological space and let $U \subseteq X$ be open and non-empty. Then a (*complex*) *uniformizing system* (also known as *orbifold chart*) *of dimension n* for U is the datum of:

- a *connected* and non-empty open set $\widetilde{U} \subseteq \mathbb{C}^n$;
- a *finite* group G of holomorphic automorphisms of $\widetilde{U}$;
- a continuous, surjective and G-invariant map $\pi : \widetilde{U} \to U$, which induces a homeomorphism between $\widetilde{U}/G$ and U, where we give to $\widetilde{U}/G$ the quotient topology.

Remark 1.4. In this work we will always assume that G *acts effectively*; the orbifolds which have this property are usually called *reduced* or *effective* (the precise definitions of orbifold and orbifold atlas will be given in the following pages). Some authors don’t use this restriction: in this case G is a priori a group together with a representation $\psi : G \to \mathrm{Aut}(\widetilde{U})$ which is not necessarily faithful.

Remark 1.5. Some articles (see, for example, [10], Def. 2.1.1) don’t require that $\widetilde{U}$ is a connected *open set* of $\mathbb{C}^n$, but only that it is a connected complex *manifold*, while all the other properties are exactly the same. This is very useful in order to define orbifold atlases for global quotients, but this definition is not equivalent to the previous one; however, it is not so difficult to prove that the 2 definitions agree when we pass to equivalence classes of orbifold atlases (see Section 1.6). We prefer to use the previous definition only because it is more common in literature.

Definition 1.6. Let $(\widetilde{U}, G, \pi)$ be a uniformizing system and let $\tilde{x} \in \widetilde{U}$. Then we define the *isotropy subgroup* (also known as *stabilizer group*) at $\tilde{x}$ as the subgroup $G_{\tilde{x}} := \{g \in G \text{ s.t. } g(\tilde{x}) = \tilde{x}\}$.

Lemma 1.7. *Let $(\widetilde{U}, G, \pi)$ be a uniformizing system, let $\tilde{x} \in \widetilde{U}$ and $g \in G \smallsetminus G_{\tilde{x}}$. Then there exists a positive radius $r = r(\tilde{x}, g)$ such that if we call B_r the open ball with radius r and centered in $\tilde{x}$, we have $g(B_r) \cap B_r = \varnothing$.*

Definition 1.8. Let us fix two uniformizing systems $(\widetilde{U}, G, \pi)$ and $(\widetilde{V}, H, \phi)$ for open sets U, V in X with $U \subseteq V$. Then a (*complex*) *embedding* λ from the first uniformizing system to the second one is given by a holomorphic embedding $\lambda : \widetilde{U} \to \widetilde{V}$ such that $\phi \circ \lambda = \pi$.

Lemma 1.9. *Let us fix a uniformizing system $(\widetilde{U}, G, \pi)$ and any point $\tilde{x}$ in $\widetilde{U}$, together with an open neighborhood $\widetilde{A}$ of it in $\widetilde{U}$. Then there exist a uniformizing system of the form $(\widetilde{U}', G', \pi')$ and an embedding λ of it into the previous one, such that*:

- *$\widetilde{U}'$ is an open connected neighborhood of $\tilde{x}$, completely contained in $\widetilde{A}$*;
- *λ is the inclusion map of $\widetilde{U}'$ in $\widetilde{U}$*;
- *G' is the set of elements of the isotropy subgroup $G_{\tilde{x}}$, restricted to $\widetilde{U}'$*;
- *up to a biholomorphic change of coordinates all the elements of G' act linearly on $\widetilde{U}'$.*

The proof of this lemma uses Lemma 1.7 for every g in the finite set $G \smallsetminus G_{\tilde{x}}$, together with Cartan's linearization lemma (see [4], Lemma 1) in order to find the open neighborhood $\widetilde{U}'$ of $\tilde{x}$ in $\widetilde{U}$; then it suffices to use the definition of quotient topology for the open set U homeomorphic to $\widetilde{U}/G$.

Definition 1.10. Let $(\widetilde{U}, G, \pi)$ be a uniformizing system for an open set U of X; then for every $g \in G \smallsetminus \{1_{\widetilde{U}}\}$ we define the sets $\widetilde{U}_g := \{\tilde{x} \in \widetilde{U}$ s.t. $g(\tilde{x}) \neq \tilde{x}\}$ and $\widetilde{U}_G := \bigcap_{g \in G \smallsetminus \{1_{\widetilde{U}}\}} \widetilde{U}_g$ i.e. the set of points of $\widetilde{U}$ with trivial stabilizer.

Now if we fix any $g \in G \smallsetminus \{1_{\widetilde{U}}\}$, we get that the map $g - 1_{\widetilde{U}} : \widetilde{U} \to \mathbb{C}^n$ is continuous, and the set $\widetilde{U}_g$ is the preimage via this function of the open set $\mathbb{C}^n \smallsetminus \{0\}$, hence $\widetilde{U}_g$ is open in $\widetilde{U}$. Moreover, it is dense in $\widetilde{U}$, indeed if it was not dense, this would imply that there exists an open subset where $g = 1_{\widetilde{U}}$; since g is holomorphic, this implies that g is the identity on all $\widetilde{U}$, so we would get a contradiction. Now every locally compact Hausdorff space is a Baire space, hence $\widetilde{U}$ is a Baire space, so every countable intersection of open dense subsets of it is still dense. In particular, $\widetilde{U}_G$ is a *finite* intersection of open dense sets, so we get:

Lemma 1.11. *The set $\widetilde{U}_G$ is open and dense in $\widetilde{U}$.*

The following is a very useful technical result. It was proved for the first time by I. Satake (see [19]) with an extra assumption, and by I. Moerdijk and D. Pronk (see [13], Appendix, Proposition A.1) in the general case for smooth orbifolds. The following is an analogous result proved with the same technique in the case of complex orbifolds.

Lemma 1.12. *Let λ and μ be two embeddings: $(\widetilde{U}, G, \pi) \to (\widetilde{V}, H, \phi)$ between uniformizing systems of the same dimension n. Then there exists a unique $h \in H$ such that $\mu = h \circ \lambda$.*

As a consequence of Lemma 1.12 we have the following corollary, whose proof is analogous to the one given in the smooth case in [1], §1.1.

Corollary 1.13. *Any embedding $\lambda : (\widetilde{U}, G, \pi) \to (\widetilde{V}, H, \phi)$ between reduced uniformizing systems induces an injective group homomorphism $\Lambda : G \to H$ such that $\lambda \circ g = \Lambda(g) \circ \lambda$ for all $g \in G$.*

Lemma 1.14. *Let $\lambda : (\widetilde{U}, G, \pi) \to (\widetilde{V}, H, \phi)$ be an embedding and let $h \in H$. If $h(\lambda(\widetilde{U})) \cap \lambda(\widetilde{U}) \neq \varnothing$, then $h(\lambda(\widetilde{U})) = \lambda(\widetilde{U})$ and h belongs to the image of the induced injective group homomorphism $\Lambda : G \to H$.*

This is proved for the smooth case, in [13], Appendix, Lemma A.2, but the proof works also in the holomorphic case, with some small changes, so we omit it.

Definition 1.15. Let X be a paracompact and second countable Hausdorff topological space; a (*complex*)*reduced orbifold atlas of dimension n* on X is a family $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i \in I}$ of reduced uniformizing systems of dimension n, together with the family of all possible embeddings $\lambda_{ij} : (\widetilde{U}_i, G_i, \pi_i) \to (\widetilde{U}_j, G_j, \pi_j)$ for all pairs of uniformizing systems of $\mathcal{U}$, such that:

(i) the family $\{\pi_i(\widetilde{U}_i)\}_{i \in I}$ is an open cover of X;
(ii) if $(\widetilde{U}_i, G_i, \pi_i)$, $(\widetilde{U}_j, G_j, \pi_j) \in \mathcal{U}$ are uniformizing systems for U_i and U_j respectively, then for every point $x \in U_i \cap U_j$ there exist an open neighborhood $U_k \subseteq U_i \cap U_j$ of x in X, a uniformizing system $(\widetilde{U}_k, G_k, \pi_k) \in \mathcal{U}$ for U_k and embeddings:

$$(\widetilde{U}_i, G_i, \pi_i) \xleftarrow{\lambda_{ki}} (\widetilde{U}_k, G_k, \pi_k) \xrightarrow{\lambda_{kj}} (\widetilde{U}_j, G_j, \pi_j). \tag{1.1}$$

Remark 1.16. The definition implies that the family $\mathcal{U}$ completely determines the set of embeddings $\{\lambda_{ij}\}_{i,j \in I}$. Therefore, with a little abuse of notation we will always write $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i \in I}$ to denote both the family of uniformizing systems and the family of embeddings between them. So every atlas can be considered as a category, with objects given by its uniformizing systems and morphisms given by embeddings between them.

Remark 1.17. Let us fix any pair of uniformizing systems $(\widetilde{U}_i, G_i, \pi_i), (\widetilde{U}_j, G_j, \pi_j) \in \mathcal{U}$ and a pair of points $\tilde{x}_i \in \widetilde{U}_i$ and $\tilde{x}_j \in \widetilde{U}_j$ such that $\pi_i(\tilde{x}_i) = \pi_j(\tilde{x}_j)$. Then by using (1.1) and eventually by composing λ_{ki} and λ_{kj} with elements of G_i and G_j respectively, without loss of generality we can assume that we have fixed a point $\tilde{x}_k \in \widetilde{U}_k$ such that the following diagram of sets and marked points is commutative:

$$\begin{tikzcd}
\tilde{x}_i \arrow[d, phantom, sloped, "\in"] & \tilde{x}_k \arrow[d, phantom, sloped, "\in"] & \tilde{x}_j \arrow[d, phantom, sloped, "\in"] \\
\widetilde{U}_i \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\curvearrowright"] & \widetilde{U}_k \arrow[l, "\lambda_{ki}"'] \arrow[r, "\lambda_{kj}"] \arrow[d, "\pi_k"] \arrow[dr, phantom, "\curvearrowright"] & \widetilde{U}_j \arrow[d, "\pi_j"] \\
U_i & U_k \arrow[l, hook'] \arrow[r, hook] \arrow[d, phantom, sloped, "\ni"] & U_j. \\
 & x &
\end{tikzcd}$$

Remark 1.18. Let us fix any point $x \in X$, let us choose $(\widetilde{U}_i, G_i, \pi_i)$ and $(\widetilde{U}_j, G_j, \pi_j)$ together with $\tilde{x}_i \in \widetilde{U}_i$, $\tilde{x}_j \in \widetilde{U}_j$ such that $\pi_i(\tilde{x}_i) = x = \pi_j(\tilde{x}_j)$. Then using Remark 1.17 and Corollary 1.13 we get injective group homomorphisms $\Lambda_{ki} : G_k \to G_i$ and $\Lambda_{kj} : G_k \to G_j$. It is easy to see that for every $g \in (G_k)_{\tilde{x}_k}$ we have $\Lambda_{ki}(g) \in (G_i)_{\tilde{x}_i}$, moreover, using Lemma 1.14 we get that for every $g \in (G_i)_{\tilde{x}}$ there exists a unique $g' \in (G_k)_{\tilde{x}_k}$ such that $\Lambda_{ki}(g') = g$. Hence we have proved that Λ_{ki} restricts to a group isomorphism from $(G_k)_{\tilde{x}_k}$ to $(G_i)_{\tilde{x}_i}$; analogously we get a group isomorphism between $(G_k)_{\tilde{x}_k}$ and $(G_j)_{\tilde{x}_j}$, so $(G_i)_{\tilde{x}_i} \simeq (G_j)_{\tilde{x}_j}$. Hence one can give a notion of local group at a point of X, *well defined up to isomorphism*.

1.3. Local liftings and compatible systems

Now our aim is to make orbifold atlases into a category, i.e. we want to define what a morphism between orbifold atlases is. In order to do that, we have first of all to define a continuous map between the underlying topological spaces, but differently from the case of morphisms between manifolds, this will not be sufficient in general. The idea to keep in mind in the following definitions is that a morphism between orbifold atlases is essentially a continuous function which can be *locally lifted to a holomorphic* function between uniformizing systems in source and target.

Remark 1.19. The following definition of morphism (compatible system) between orbifold atlases is *almost* the usual one of good/strong map between orbifolds (see, for example, [5], §4.1) with one important difference. Indeed it is quite evident from the following definitions that after passing to equivalence classes of atlases (see Section 1.6) our definition coincides with the definition of strong/good map. We are forced to use this definition, instead of the usual one, since the objects we are dealing with are orbifold atlases and not *classes of equivalence of* orbifold atlases.

Definition 1.20. Let $\mathcal{U}$ and $\mathcal{V}$ be atlases for X and Y respectively and let $U \subseteq X$ and $V \subseteq Y$ be open sets with uniformizing systems $(\widetilde{U}, G, \pi) \in \mathcal{U}$ and $(\widetilde{V}, H, \phi) \in \mathcal{V}$ respectively. Let $f : U \to V$ be a continuous function; then a *lifting of* f from $(\widetilde{U}, G, \pi)$ to $(\widetilde{V}, H, \phi)$ is a holomorphic function $\tilde{f}_{\widetilde{U},\widetilde{V}} : \widetilde{U} \to \widetilde{V}$ such that:

$$\phi \circ \tilde{f}_{\widetilde{U},\widetilde{V}} = f \circ \pi. \tag{1.2}$$

Definition 1.21. Let $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i \in I}$ and $\mathcal{V} = \{(\widetilde{V}_j, H_j, \phi_j)\}_{j \in J}$ be atlases (not necessarily of the same dimension) for X and Y respectively and let $f : X \to Y$ be a continuous map. Then a *compatible system* for f is the datum of:

(1) a *functor* $\tilde{f} : \mathcal{U} \to \mathcal{V}$ between the associated categories (see Remark 1.16) such that if we call $(\widetilde{V}_i, H_i, \phi_i) \in \mathcal{V}$ the image of any element $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ via $\tilde{f}$, we have $f(\pi_i(\widetilde{U}_i)) \subseteq \phi_i(\widetilde{V}_i)$;
(2) a collection $\{\tilde{f}_{\widetilde{U}_i,\widetilde{V}_i}\}_{(\widetilde{U}_i,G_i,\pi_i) \in \mathcal{U}}$ where for every $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ we have that $\tilde{f}_{\widetilde{U}_i,\widetilde{V}_i}$ is a lifting for the continuous function $f|_{U_i} : U_i \to f(U_i) \subseteq V_i$ from $(\widetilde{U}_i, G_i, \pi_i)$ to $(\widetilde{V}_i, H_i, \phi_i)$;

such that for every embedding λ_{ij} from $(\widetilde{U}_i, G_i, \pi_i)$ to $(\widetilde{U}_j, G_j, \pi_j)$ in $\mathcal{U}$ we have:

$$\tilde{f}_{\widetilde{U}_j,\widetilde{V}_j} \circ \lambda_{ij} = \tilde{f}(\lambda_{ij}) \circ \tilde{f}_{\widetilde{U}_i,\widetilde{V}_i} \tag{1.3}$$

i.e. we are in the following situation:

$$\begin{tikzcd}
\widetilde{U}_i \arrow[rr, "\tilde{f}_{\widetilde{U}_i,\widetilde{V}_i}"] \arrow[dr, "\lambda_{ij}"] \arrow[dd, "\pi_i"'] & & \widetilde{V}_i \arrow[dr, "\tilde{f}(\lambda_{ij})"] \arrow[dd, "\phi_i"] & \\
& \widetilde{U}_j \arrow[rr, "\tilde{f}_{\widetilde{U}_j,\widetilde{V}_j}"] \arrow[dd, "\pi_j"'] & & \widetilde{V}_j \arrow[dd, "\phi_j"] \\
U_i \arrow[rr, "f_{|U_i}"'] \arrow[dr, hook] & & V_i \arrow[dr, hook] & \\
& U_j \arrow[rr, "f_{|U_j}"'] & & V_j
\end{tikzcd}$$

where *all the faces of the cube are commutative*; indeed:

- the lower face is commutative because we are just restricting the function f from U_j to U_i;
- the left and right sides are commutative by definition of embedding in $\mathcal{U}$ and $\mathcal{V}$ respectively;
- the front and back sides are both commutatives because of (1.2);
- the top side is commutative because of (1.3).

With a little abuse of notation, we will always write $\tilde{f}:\mathcal{U}\to\mathcal{V}$ to denote a compatible system for f, i.e. with $\tilde{f}$ we will usually mean not only the functor which satisfies (1), but also the collection of local liftings described in (2).

Remark 1.22. Note that in (1) *it is sufficient to require that $\tilde{f}$ preserves compositions* (and actually, this is the standard definition in most articles). Indeed, suppose that we have fixed any $i\in I$ and let us call $h:=\tilde{f}(1_{\widetilde{U}_i})\in H_i$; then we have $h=\tilde{f}(1_{\widetilde{U}_i})=\tilde{f}(1^2_{\widetilde{U}_i})=\tilde{f}(1_{\widetilde{U}_i})^2=h^2$; since H_i is a group by hypothesis, this implies that $h=1_{\widetilde{V}_i}$, i.e. $\tilde{f}$ preserves all the identities.

Definition 1.23. Now let us consider 3 fixed orbifold atlases $\mathcal{U},\mathcal{V},\mathcal{W}$ for X, Y and Z respectively, together with 2 continuous functions $f:X\to Y$, $g:Y\to Z$ and compatible systems $\tilde{f}:\mathcal{U}\to\mathcal{V}$ and $\tilde{g}:\mathcal{V}\to\mathcal{W}$. For every uniformizing system $(\widetilde{U}_i,G_i,\pi_i)\in\mathcal{U}$, let us call:

$$(\widetilde{V}_i,H_i,\phi_i):=\tilde{f}(\widetilde{U}_i,G_i,\pi_i)\quad\text{and}\quad(\widetilde{W}_i,K_i,\xi_i):=\tilde{g}(\widetilde{V}_i,H_i,\phi_i).$$

Then we define the compatible system $\tilde{g}\circ\tilde{f}$ for $g\circ f$ as the functor $\tilde{g}\circ\tilde{f}:\mathcal{U}\to\mathcal{W}$ together with the collection of liftings:

$$\big\{(\tilde{g}\circ\tilde{f})_{\widetilde{U}_i,\widetilde{W}_i}:=\tilde{g}_{\widetilde{V}_i,\widetilde{W}_i}\circ\tilde{f}_{\widetilde{U}_i,\widetilde{V}_i}\big\}_{(\widetilde{U}_i,G_i,\pi_i)\in\mathcal{U}}.$$

1.4. Natural transformations between compatible systems

With the previous definitions we get a category, but we recall that we wanted to make orbifold atlases into a 2-category, so we give the following definition (a slight change of Def. 1.3.6 in [15], which I think was too much restrictive for our purposes).

Definition 1.24. Let us fix atlases $\mathcal{U}$ and $\mathcal{V}$ for X and Y respectively and let $\tilde{f}_1,\tilde{f}_2:\mathcal{U}\to\mathcal{V}$ be compatible systems for *the same* continuous map $f:X\to Y$. For simplicity, for every uniformizing system $(\widetilde{U}_i,G_i,\pi_i)\in\mathcal{U}$ and for every embedding λ_{ij}, let us call:

$$\big(\widetilde{V}^m_i,H^m_i,\phi^m_i\big):=\tilde{f}_m(\widetilde{U}_i,G_i,\pi_i)\quad\text{and}\quad\lambda^m_{ij}:=\tilde{f}_m(\lambda_{ij})\quad\text{for } m=1,2.$$

Then a *natural transformation of compatible systems from $\tilde{f}_1$ to $\tilde{f}_2$* is a family:

$$\big\{\delta_{\widetilde{U}_i}=\delta_{(\widetilde{U}_i,G_i,\pi_i)}:\big(\widetilde{V}^1_i,H^1_i,\phi^1_i\big)\to\big(\widetilde{V}^2_i,H^2_i,\phi^2_i\big)\big\}_{(\widetilde{U}_i,G_i,\pi_i)\in\mathcal{U}}$$

of embeddings in $\mathcal{V}$, such that:

(i) for every $(\widetilde{U}_i,G_i,\pi_i)\in\mathcal{U}$ we have $(\tilde{f}_2)_{\widetilde{U}_i,\widetilde{V}^2_i}=\delta_{\widetilde{U}_i}\circ(\tilde{f}_1)_{\widetilde{U}_i,\widetilde{V}^1_i}$;
(ii) for every embedding λ_{ij} in $\mathcal{U}$ we have a commutative diagram in $\mathcal{V}$:

$$\begin{tikzcd}
(\widetilde{V}^1_i, H^1_i, \phi^1_i) \arrow[r, "\delta_{\widetilde{U}_i}"] \arrow[d, "\lambda^1_{ij}"'] \arrow[dr, phantom, "\curvearrowright"] & (\widetilde{V}^2_i, H^2_i, \phi^2_i) \arrow[d, "\lambda^2_{ij}"] \\
(\widetilde{V}^1_j, H^1_j, \phi^1_j) \arrow[r, "\delta_{\widetilde{U}_j}"'] & (\widetilde{V}^2_j, H^2_j, \phi^2_j).
\end{tikzcd} \tag{1.4}$$

Whenever we have a natural transformation as before, we will denote it as $\delta : \tilde{f}_1 \Rightarrow \tilde{f}_2$. Note that if we ignore the additional properties of the compatible systems $\tilde{f}_1$ and $\tilde{f}_2$ and we consider them just as functors, we get that condition (ii) is just the description of a natural transformation from the functor $\tilde{f}_1$ to the functor $\tilde{f}_2$ (so the following horizontal and vertical compositions of natural transformations will be modeled on the corresponding constructions described, for example, in [2]).

Definition 1.25. Let us fix orbifold atlases $\mathcal{U}$ for X and $\mathcal{V}$ for Y, a continuous map $f : X \to Y$, 3 compatible systems $\tilde{f}_m : \mathcal{U} \to \mathcal{V}$ for $m = 1, 2, 3$ and natural transformations $\delta : \tilde{f}_1 \Rightarrow \tilde{f}_2$ and $\sigma : \tilde{f}_2 \Rightarrow \tilde{f}_3$. Then we define the *vertical composition* $\sigma \odot \delta : \tilde{f}_1 \Rightarrow \tilde{f}_3$ as follows: for any $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ we set $(\sigma \odot \delta)_{\widetilde{U}_i} := \sigma_{\widetilde{U}_i} \circ \delta_{\widetilde{U}_i}$, which is an embedding in $\mathcal{V}$ between the images of $(\widetilde{U}_i, G_i, \pi_i)$ via $\tilde{f}_1$ and $\tilde{f}_3$ respectively.

A direct check proves that properties (i) and (ii) of Definition 1.24 are satisfied, hence $\sigma \odot \delta$ is actually a natural transformation from $\tilde{f}_1$ to $\tilde{f}_3$.

Definition 1.26. For every uniformizing system $\tilde{f} : \mathcal{U} \to \mathcal{V}$, we define the natural transformation $i_{\tilde{f}}$ as follows: for any uniformizing system $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ we set as usual $(\widetilde{V}_i, H_i, \phi_i) := \tilde{f}(\widetilde{U}_i, G_i, \pi_i)$ and we define $(i_{\tilde{f}})_{\widetilde{U}_i} := 1_{\widetilde{V}_i}$. Clearly $i_{\tilde{f}}$ is a natural transformation from $\tilde{f}$ to itself; moreover, for any $\alpha : \tilde{f} \Rightarrow \tilde{g}$ and for any $\beta : \tilde{h} \Rightarrow \tilde{f}$ we have:

$$\alpha \odot i_{\tilde{f}} = \alpha \quad \text{and} \quad i_{\tilde{f}} \odot \beta = \beta. \tag{1.5}$$

Definition 1.27. Let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be orbifold atlases for X, Y and Z respectively; let $\tilde{f}_m$ and $\tilde{g}_m$ be compatible systems for $f : X \to Y$ and $g : Y \to Z$ respectively, for $m = 1, 2$. Moreover, assume that we have natural transformations $\delta : \tilde{f}_1 \Rightarrow \tilde{f}_2$ and $\eta : \tilde{g}_1 \Rightarrow \tilde{g}_2$. Then we define a *horizontal composition* $\eta * \delta : (\tilde{g}_1 \circ \tilde{f}_1) \Rightarrow (\tilde{g}_2 \circ \tilde{f}_2)$ as follows: for any $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ we set $(\eta * \delta)_{\widetilde{U}_i} := \eta_{\widetilde{V}^2_i} \circ \tilde{g}_1(\delta_{\widetilde{U}_i})$.

Every map of this form is actually an embedding between uniformizing systems because composition of embeddings: indeed $\eta_{\widetilde{V}^2_i}$ is so by definition and $\tilde{g}_1(\delta_{\widetilde{U}_i})$ is an embedding because the functor $\tilde{g}_1$ maps embeddings to embeddings. Again a very simple check proves that properties (i) and (ii) of Definition 1.24 are satisfied.

1.5. The 2-category (**Pre-Orb**)

Proposition 1.28. *The definitions of orbifold atlases, compatible systems, natural transformations and compositions* $\circ, \odot, *$ *give rise to a* 2*-category, that we will denote with* (**Pre-Orb**).

Note that we cannot call this 2-category (**Orb**) because we will see in Section 1.6 that orbifolds are equivalence classes of orbifold atlases.

Proof. In order to construct a 2-category, we have to define data (1)–(4) and to verify axioms (a)–(d) of Definition 1.1.

(1) First of all, the class of objects is just the set of all orbifold atlases for every topological space X (if any).
(2) If $\mathcal{U}$ and $\mathcal{V}$ are atlases over X and Y, we define a small category (**Pre-Orb**)$(\mathcal{U}, \mathcal{V})$ as follows: the space of objects is the set of all compatible systems $\tilde{f} : \mathcal{U} \to \mathcal{V}$ (if any) for all continuous maps $f : X \to Y$; for any pair of compatible systems $\tilde{f}$ and $\tilde{g}$ for f and g respectively, we define:

$$(\mathbf{Pre\text{-}Orb})(\mathcal{U}, \mathcal{V})(\tilde{f}, \tilde{g}) := \begin{cases} \text{natural transformations } \tilde{f} \Rightarrow \tilde{g} & \text{if } f = g, \\ \varnothing & \text{otherwise.} \end{cases}$$

The vertical composition $\odot$ is clearly associative; moreover using (1.5) we get that the identity over any object $\tilde{f}$ is just $i_{\tilde{f}}$.

(3) For every triple $\mathcal{U}, \mathcal{V}, \mathcal{W}$ of objects, we define the functor "composition" as $\circ$ on pairs of composable morphisms and as $*$ on pairs of composable 2-morphisms.
We want to prove that this gives rise to a functor. It is easy to see that identities are preserved, so let us only prove that it preserves compositions. Let us fix any diagram of the form:

$$\begin{tikzcd} \mathcal{U} \arrow[r, bend left=60, "\tilde{f}_1"] \arrow[r, "\tilde{f}_2"'] \arrow[r, bend right=60, "\tilde{f}_3"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\delta"] \arrow[r, phantom, shift right=3ex, "\Downarrow\sigma"] & \mathcal{V} \arrow[r, bend left=60, "\tilde{g}_1"] \arrow[r, "\tilde{g}_2"'] \arrow[r, bend right=60, "\tilde{g}_3"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\eta"] \arrow[r, phantom, shift right=3ex, "\Downarrow\mu"] & \mathcal{W}. \end{tikzcd}$$

To say that compositions are preserved is equivalent to prove that $(\mu \odot \eta) * (\sigma \odot \delta) \stackrel{?}{=} (\mu * \sigma) \odot (\eta * \delta)$. In other words, we have to prove that the *interchange law* (see [2], Proposition 1.3.5) is satisfied. So let us verify that this last identity is true: for any uniformizing system $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ we have:

$$\begin{aligned} \big((\mu \odot \eta) * (\sigma \odot \delta)\big)_{\widetilde{U}_i} &= (\mu \odot \eta)_{\widetilde{V}_i^3} \circ \tilde{g}_1\big((\sigma \odot \delta)_{\widetilde{U}_i}\big) \\ &= \mu_{\widetilde{V}_i^3} \circ \eta_{\widetilde{V}_i^3} \circ \tilde{g}_1(\sigma_{\widetilde{U}_i}) \circ \tilde{g}_1(\delta_{\widetilde{U}_i}) \stackrel{*}{=} \mu_{\widetilde{V}_i^3} \circ \tilde{g}_2(\sigma_{\widetilde{U}_i}) \circ \eta_{\widetilde{V}_i^2} \circ \tilde{g}_1(\delta_{\widetilde{U}_i}) \\ &= (\mu * \sigma)_{\widetilde{U}_i} \odot (\eta * \delta)_{\widetilde{U}_i} = \big((\mu * \sigma) \odot (\eta * \delta)\big)_{\widetilde{U}_i} \end{aligned}$$

where the passage denoted with $\stackrel{*}{=}$ is just (1.4) for the natural transformation $\eta : \tilde{g}_1 \Rightarrow \tilde{g}_2$ and for the embedding $\lambda := \sigma_{\widetilde{U}_i}$.

(4) It remains to define the "identities" of (**Pre-Orb**), so for every atlas $\mathcal{U}$ we define $1_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}$ to be a compatible system over the identity on X, described as the identity functor from the category associated to $\mathcal{U}$ to itself, together with the collection of liftings for the identity map on X given by $\{(1_{\mathcal{U}})_{\widetilde{U}_i, \widetilde{U}_i} := 1_{\widetilde{U}_i}\}_{(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}}$; moreover, we define $i_{\mathcal{U}}$ as the natural transformation $i_{1_{\mathcal{U}}}$.

So we have defined all the data of a 2-category; the axioms (a), (c) and (d) are easy to verify, so we omit them. We only verify axiom (b), so let us fix any diagram of compatible systems and natural transformations of the form:

$$\begin{tikzcd} \mathcal{U} \arrow[r, bend left, "\tilde{f}_1"] \arrow[r, bend right, "\tilde{f}_2"'] \arrow[r, phantom, "\Downarrow\delta"] & \mathcal{V} \arrow[r, bend left, "\tilde{g}_1"] \arrow[r, bend right, "\tilde{g}_2"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{W} \arrow[r, bend left, "\tilde{h}_1"] \arrow[r, bend right, "\tilde{h}_2"'] \arrow[r, phantom, "\Downarrow\omega"] & \mathcal{Z} \end{tikzcd}$$

and any uniformizing system $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$; then let us define:

$$\big(\widetilde{W}_i^{mn}, K_i^{mn}, \xi_i^{mn}\big) := \tilde{g}_n \circ \tilde{f}_m(\widetilde{U}_i, G_i, \pi_i) \quad \text{for } m, n = 1, 2.$$

If we use also the notation introduced in Definition 1.24 we have:

$$\begin{aligned} \big((\omega * \eta) * \delta\big)_{\widetilde{U}_i} &= (\omega * \eta)_{\widetilde{V}_i^2} \circ \big((\tilde{h}_1 \circ \tilde{g}_1)(\delta_{\widetilde{U}_i})\big) \\ &= \omega_{\widetilde{W}_i^{22}} \circ \tilde{h}_1(\eta_{\widetilde{V}_i^2}) \circ \big(\tilde{h}_1 \circ \tilde{g}_1(\delta_{\widetilde{U}_i})\big) = \omega_{\widetilde{W}_i^{22}} \circ \tilde{h}_1\big(\eta_{\widetilde{V}_i^2} \circ \tilde{g}_1(\delta_{\widetilde{U}_i})\big) \\ &= \omega_{\widetilde{W}_i^{22}} \circ \tilde{h}_1\big((\eta * \delta)_{\widetilde{U}_i}\big) = \big(\omega * (\eta * \delta)\big)_{\widetilde{U}_i}. \end{aligned}$$

Hence we have proved that $(\omega * \eta) * \delta = \omega * (\eta * \delta)$. $\square$

1.6. Equivalent orbifold atlases

It is well known that a manifold is an equivalence class of compatible manifold atlases; in literature there is an analogous notion in the framework of orbifolds.

Definition 1.29. (Equivalent to [10], §2.1 and [5], Def. 4.1.2.) Two atlases $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i \in I}$ and $\mathcal{V} = \{(\widetilde{V}_j, H_j, \phi_j)\}_{j \in J}$ on the same space X are *equivalent at a point* $x \in X$ if there exists a uniformizing system $(\widetilde{W}, K, \xi)$ around x, together with 2 embeddings:

$$(\widetilde{U}_i, G_i, \pi_i) \xleftarrow{\lambda} (\widetilde{W}, K, \xi) \xrightarrow{\lambda'} (\widetilde{V}_j, H_j, \phi_j)$$

for some uniformizing systems $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ and $(\widetilde{V}_j, H_j, \phi_j) \in \mathcal{V}$. Note that we don't require that $(\widetilde{W}, K, \xi)$ belongs to $\mathcal{U}$ and/or $\mathcal{V}$. Two atlases on a space X are *equivalent* iff they are equivalent at every point of X.

Lemma 1.30. *This is an equivalence relation* (*see Appendix* A *for the proof*).

Actually, in literature one can also find these definitions:

Definition 1.31. ([1], §1) An orbifold atlas $\mathcal{U}$ on X is said to *refine* another orbifold atlas $\mathcal{V}$ on the same topological space if for every uniformizing system in $\mathcal{U}$ there exists an embedding of it into some uniformizing system of $\mathcal{V}$. Equivalently, $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i\in I}$ is a refinement of $\mathcal{V} = \{(\widetilde{V}_j, H_j, \phi_j)\}_{j\in J}$ iff there exist a set map $\gamma : I \to J$ and embeddings $\lambda_i : (\widetilde{U}_i, G_i, \pi_i) \to (\widetilde{V}_{\gamma(i)}, H_{\gamma(i)}, \phi_{\gamma(i)})$ for every $i \in I$. In a some sense, we can consider this as a compatible system $\mathcal{U} \to \mathcal{V}$ for the identity on X, except for the fact that in general this will not be a functor (actually, it is not even defined on embeddings). However, we will use the same abuse of notation that we used for compatible system, i.e. we will write $(\widetilde{V}_i, H_i, \phi_i)$ instead of $(\widetilde{V}_{\gamma(i)}, H_{\gamma(i)}, \phi_{\gamma(i)})$.

Definition 1.32. ([1], §1) Two orbifold atlases on the same space X are *equivalent* if they have a common refinement.

Proposition 1.33. *Definitions* 1.29 *and* 1.32 *coincide.*

Proof. Let us fix a space X and two orbifold atlases $\mathcal{U}_1$ and $\mathcal{U}_2$ which are equivalent with respect to Definition 1.29; then we define a family $\mathcal{W}$ whose elements are *all* the uniformizing systems (for some open set of X) that have an embedding in at least one chart of $\mathcal{U}_1$ and one chart of $\mathcal{U}_2$. Then we consider $\mathcal{W}$ as an orbifold atlas by adding all the possible embeddings between its charts and it is easy to see that actually $\mathcal{W}$ is an orbifold atlas on X and it refines both $\mathcal{U}_1$ and $\mathcal{U}_2$.

Conversely, if there exists a common refinement $\mathcal{W}$ of two atlases $\mathcal{U}_1$ and $\mathcal{U}_2$, then we get that they are equivalent with respect to Definition 1.29 by a direct application of Definition 1.31. □

Corollary 1.34. *The notion of having a common refinement is a relation of equivalence.*

So it makes sense to give the following definition:

Definition 1.35. ([1], Def. 1.2) A *complex orbifold structure* on a second countable paracompact Hausdorff topological space X is an equivalence class of orbifold atlases on X. We will denote such an object by $\mathcal{X}$ or $[X]$. We will call *orbifold* the pair $(X, \mathcal{X})$, or, by abuse of notation, just the orbifold structure $\mathcal{X}$. We say that $\mathcal{X}$ has *dimension n* if there is an atlas $\mathcal{U}$ of dimension n in the class $\mathcal{X}$. That is equivalent to say that *every* atlas of the class has the same dimension n.

Definition 1.36. For every orbifold $\mathcal{X}$ on X we define the *maximal atlas* associated to it as the family of all the uniformizing systems of all the atlases of the class $\mathcal{X}$. If one considers also all the possible embeddings between the charts of this family, one can easily prove that it is actually an orbifold atlas for X, that it belongs to the class $\mathcal{X}$ and that it is refined by every atlas of $\mathcal{X}$.

At the end of this paper we will also need the following new equivalence relation on the set of orbifold atlases.

Definition 1.37. Suppose that we have an orbifold atlas $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i\in I}$ on X and a homeomorphism $\varphi : X' \to X$; then we can define the following family:

$$\varphi^*(\mathcal{U}) := \left\{\left(\widetilde{U}_i, G_i, \varphi^{-1} \circ \pi_i\right)\right\}_{i\in I}$$

which is an orbifold atlas on X'. Now if we have two orbifold atlases $\mathcal{U}$ on X and $\mathcal{U}'$ on X', we say that they are *equivalent* if the following 2 conditions hold:

(a) there exists a homeomorphism $\varphi : X' \to X$;
(b) the orbifold atlases $\varphi^*(\mathcal{U})$ and $\mathcal{U}'$ on X' are equivalent with respect to the previous definition.

This relation is obviously reflexive and symmetric; moreover, one can easily prove transitivity using the transitivity of the equivalence relation on a fixed topological space. Hence *it is an equivalence relation on the set of all orbifold atlases over any topological space, i.e. over the objects of* (**Pre-Orb**).

2. Internal groupoids in a category $\mathscr{C}$

We will use this section in order to recall the standard literature about internal groupoids in any category $\mathscr{C}$ and then we will specialize to $\mathscr{C} =$ (**Manifolds**).

2.1. Groupoid objects in a fixed category

Definition 2.1. ([3], Def. A.3.4) A *groupoid object* or *internal groupoid* in a category $\mathscr{C}$ is the datum of two objects R, U and five morphisms of $\mathscr{C}$:

- $s, t : R \rightrightarrows U$ such that the fiber product $R_t \times_s R$ *exists* in $\mathscr{C}$; these two maps are usually called *source* and *target* of the groupoid object;
- $m : R_t \times_s R \to R$, called *multiplication*;
- $i : R \to R$, known as *inverse* of the groupoid object;
- $e : U \to R$, called *identity*;

which satisfy the following axioms:

(i) $s \circ e = 1_U = t \circ e$;
(ii) if we call pr_1 and pr_2 the two projections from the fibered product $R_t \times_s R$ to R, then we have $s \circ m = s \circ pr_1$ and $t \circ m = t \circ pr_2$;
(iii) (associativity) the two morphisms $m \circ (1_R \times m)$ and $m \circ (m \times 1_R)$ from $R_t \times_s R_t \times_s R$ to R are equal;
(iv) (unit) the two morphisms $m \circ (e \circ s, 1_R)$ and $m \circ (1_R, e \circ t)$ from R to R are both equal to the identity of R;
(v) (inverse) $i \circ i = 1_R$, $s \circ i = t$ (and therefore $t \circ i = s$). Moreover, we require that $m \circ (1_R, i) = e \circ s$ and $m \circ (i, 1_R) = e \circ t$:

If we assume that the category $\mathscr{C}$ is fixed, we will denote any groupoid object as before by $R \underset{t}{\overset{s}{\rightrightarrows}} U$. In some articles one can also find the notation (U, R, s, t, m, e, i) or $R_s \times_t R \xrightarrow{m} R \xrightarrow{i} R \underset{t}{\overset{s}{\rightrightarrows}} U \xrightarrow{e} R$.

2.2. Morfisms and 2-morphisms between groupoid objects

Definition 2.2. ([11], §2.1) Given two groupoid objects $R \underset{t}{\overset{s}{\rightrightarrows}} U$ and $R' \underset{t'}{\overset{s'}{\rightrightarrows}} U'$ in a fixed category $\mathscr{C}$, a *morphism* between them is a pair (ψ, Ψ), where $\psi : U \to U'$ and $\Psi : R \to R'$ are both *morphisms in* $\mathscr{C}$, which together commute with all structure morphisms of the two groupoid objects. In other words, we ask that the following five identities are satisfied:

$$s' \circ \Psi = \psi \circ s, \qquad t' \circ \Psi = \psi \circ t, \qquad \Psi \circ e = e' \circ \psi, \tag{2.1}$$

$$\Psi \circ m = m' \circ (\Psi \times \Psi) \quad \text{and} \quad \Psi \circ i = i' \circ \Psi. \tag{2.2}$$

Remark 2.3. Using axiom (i) of the previous definition and (2.1), we get that $\psi = s' \circ \Psi \circ e$.

Definition 2.4. Let us consider 3 groupoid objects in $\mathscr{C}$ and 2 morphisms:

$$\left(R \underset{t}{\overset{s}{\rightrightarrows}} U\right) \xrightarrow{(\psi,\Psi)} \left(R' \underset{t'}{\overset{s'}{\rightrightarrows}} U'\right) \xrightarrow{(\phi,\Phi)} \left(R'' \underset{t''}{\overset{s''}{\rightrightarrows}} U''\right).$$

It is easy to see that if we define the *composition* $(\phi, \Phi) \circ (\psi, \Psi)$ as $(\phi \circ \psi, \Phi \circ \Psi)$, then this is again a morphism of groupoid objects from $(R \underset{t}{\overset{s}{\rightrightarrows}} U)$ to $(R'' \underset{t''}{\overset{s''}{\rightrightarrows}} U'')$.

Now we want to make groupoid objects into a 2-category, i.e. we want to define 2-morphisms, which will be called "natural transformations".

Definition 2.5. ([18], Def. 2.3) Suppose we have fixed two morphisms of groupoid objects $(\psi, \Psi), (\phi, \Phi) : (R \underset{t}{\overset{s}{\rightrightarrows}} U) \to (R' \underset{t'}{\overset{s'}{\rightrightarrows}} U')$ in $\mathscr{C}$. Then a *natural transformation* $\alpha : (\psi, \Psi) \Rightarrow (\phi, \Phi)$ is the datum of a *morphism* $\alpha : U \to R'$ in $\mathscr{C}$ such that the following 2 conditions hold:

(i) $s' \circ \alpha = \psi$ and $t' \circ \alpha = \phi$;
(ii) $m' \circ (\alpha \circ s, \Phi) = m' \circ (\Psi, \alpha \circ t)$.

Note that using (i) together with the definition of morphism between groupoid objects, we get that $t' \circ (\alpha \circ s) = \phi \circ s = s' \circ \Phi$ and $t' \circ \Psi = \psi \circ t = s' \circ (\alpha \circ t)$; hence we can consider both $(\alpha \circ s, \Phi)$ and $(\Psi, \alpha \circ t)$ as morphisms in the category $\mathscr{C}$ from R to $R'{}_{t'} \times_{s'} R'$, so the 2 sides of (ii) are both well defined.

Definition 2.6. Let us consider any diagram as follows:

$$\begin{tikzcd}
(R \overset{s}{\underset{t}{\rightrightarrows}} U) \arrow[rr, bend left=50, "{(\psi_1,\Psi_1)}"] \arrow[rr, "{(\psi_2,\Psi_2)}"'] \arrow[rr, bend right=50, "{(\psi_3,\Psi_3)}"'] \arrow[rr, phantom, shift left=4ex, "\Downarrow\alpha"] \arrow[rr, phantom, shift right=4ex, "\Downarrow\beta"] & & (R' \overset{s'}{\underset{t'}{\rightrightarrows}} U');
\end{tikzcd}$$

using Definition 2.5 for α and β we get that it makes sense to consider the morphism $(\alpha,\beta): U \to R'{}_{t'} \times_{s'} R'$ and we can define $\beta \odot \alpha := m' \circ (\alpha,\beta): U \to R'$. A direct calculation proves that this is a natural transformation, so we call it *vertical composition* of α and β and we denote it with $\beta \odot \alpha : (\psi_1,\Psi_1) \Rightarrow (\psi_3,\Psi_3)$.

Definition 2.7. For every morphism $(\psi,\Psi): (R \overset{s}{\underset{t}{\rightrightarrows}} U) \to (R' \overset{s'}{\underset{t'}{\rightrightarrows}} U')$ we define its identity as the natural transformation $i_{(\psi,\Psi)} := e' \circ \psi = \Psi \circ e : (\psi,\Psi) \Rightarrow (\psi,\Psi)$. A simple check proves that this is actually a natural transformation and that for every $\alpha : (\psi,\Psi) \Rightarrow (\phi,\Phi)$ and for every $\beta : (\theta,\Theta) \Rightarrow (\psi,\Psi)$ we have $\alpha \odot i_{(\psi,\Psi)} = \alpha$ and $i_{(\psi,\Psi)} \odot \beta = \beta$.

Definition 2.8. Let us consider any diagram of the form:

$$\begin{tikzcd}
(R \overset{s}{\underset{t}{\rightrightarrows}} U) \arrow[r, bend left, "{(\psi_1,\Psi_1)}"] \arrow[r, bend right, "{(\psi_2,\Psi_2)}"'] \arrow[r, phantom, "\Downarrow\alpha"] & (R' \overset{s'}{\underset{t'}{\rightrightarrows}} U') \arrow[r, bend left, "{(\phi_1,\Phi_1)}"] \arrow[r, bend right, "{(\phi_2,\Phi_2)}"'] \arrow[r, phantom, "\Downarrow\beta"] & (R'' \overset{s''}{\underset{t''}{\rightrightarrows}} U'').
\end{tikzcd} \tag{2.3}$$

In particular, we get that:

$$s' \circ \alpha = \psi_1 \quad \text{and} \quad t' \circ \alpha = \psi_2; \tag{2.4}$$

$$s'' \circ \beta = \phi_1 \quad \text{and} \quad t'' \circ \beta = \phi_2; \tag{2.5}$$

$$t'' \circ (\Phi_1 \circ \alpha) = \phi_1 \circ t' \circ \alpha \overset{(2.4)}{=} \phi_1 \circ \psi_2 \overset{(2.5)}{=} s'' \circ (\beta \circ \psi_2). \tag{2.6}$$

Hence we have a morphism $(\Phi_1 \circ \alpha, \beta \circ \psi_2): U \to R''{}_{t''} \times_{s''} R''$, so we can set:

$$\beta * \alpha := m'' \circ (\Phi_1 \circ \alpha, \beta \circ \psi_2): U \to R''.$$

A long diagram chasing proves that $\beta * \alpha$ is a natural transformation from $(\phi_1,\Phi_1) \circ (\psi_1,\Psi_1)$ to $(\phi_2,\Phi_2) \circ (\psi_2,\Psi_2)$, so we call it *horizontal composition* of α and β.

Using all the previous data it is not so difficult to prove the following result, which is quite well known in the literature:

Proposition 2.9. *Let us fix a category $\mathscr{C}$; the definitions of groupoid objects, morphisms of groupoid objects, natural transformations and compositions $\circ, \odot, *$ give rise to a 2-category, that we will denote with ($\mathscr{C}$-**Groupoids**).*

We omit this proof, since this is just a direct check of all the axioms of Definition 1.1. In any case, we will just be interested in a special case of this result, that will be recalled below.

2.3. The 2-category (**Grp**)

Given any pair of morphisms $s,t: R \to U$ in a fixed category $\mathscr{C}$, in order to define a groupoid object from this data, we must be sure that the fibered product $R_t \times_s R$ exists. This is always ensured e.g. when we work in the categories (**Sets**), (**Groups**) or (**Schemes**), but in general it is no more true in the category (**Manifolds**) (complex manifolds and holomorphic maps between them). In this last category (which will be used in this section) fiber products exist only if we put some additional hypothesis on f and/or g. One of the most useful conditions is about submersions:

Proposition 2.10. *Let us fix any pair of holomorphic maps between complex manifolds $f: X \to Y$ and $g: Z \to Y$. If f is a submersion, then the set-theoretical fiber product*:

$$X \times_Y Z = \{(x,z) \in X \times Z \text{ s.t. } f(x) = g(z)\}$$

is a complex submanifold of $X \times Z$*, with complex dimension equal to* $dim(X) + dim(Z) - dim(Y)$*. Moreover, the map* $pr_2 : X \times_Y Z \to Z$ *is again a submersion and if* f *is étale* (*i.e. a local biholomorphism*)*, so is* pr_2.

For a hint of the proof, see [7], Chapter IV.1, Exercise 6 and (for the smooth case) [6], Chapter XVI.8, Exercise 10.

Definition 2.11. (Adapted from [9], Def. 2.11.) A groupoid object in (**Manifolds**) is called *Lie groupoid* if both s and t are holomorphic *submersions*.

Remark 2.12. Using the previous proposition, we get that the fiber product used in the first point of the definition of groupoid objects exists if we work with Lie groupoids. Moreover, the resulting maps pr_1 and pr_2 are again both submersions. Hence by induction it is easy to prove that in this case there exist fiber products of the form $R_t \times_s \cdots {}_t \times_s R$ for finitely many terms.

Definition 2.13. ([11], §1.5) A groupoid object in (**Manifolds**) is *proper* if the map $(s,t) : R \to U \times U$ (called *relative diagonal*) is proper, i.e. if the preimage of any compact set in $U \times U$ is compact in R.

Definition 2.14. ([11], §1.2) An *étale groupoid* is a groupoid object $R \overset{s}{\underset{t}{\rightrightarrows}} U$ in (**Manifolds**) such that the maps s and t are both étale (i.e. local biholomorphisms). Using Remark 2.12, the fiber products $R_t \times_s R$ and $R_t \times_s R\,{}_t \times_s R$ exist, so all Definition 2.1 still makes sense (clearly every étale groupoid is also a Lie groupoid).

Definition 2.15. ([11], Example 1.5) Let $R \overset{s}{\underset{t}{\rightrightarrows}} U$ be an étale proper groupoid, let us fix any point $\tilde{x} \in U$ and let us define $R_{\tilde{x}} := (s,t)^{-1}\{(\tilde{x},\tilde{x})\}$, called the *isotropy subgroup of* $\tilde{x}$. This set is naturally a group (using the multiplication m) and it is compact because (s,t) is proper; moreover its points are all isolated because s is étale, hence locally invertible; hence $R_{\tilde{x}}$ is a *finite* set of points of R. Since both s and t are étale, for every point g in this set, we can find a sufficiently small open neighborhood W_g of g where both s and t are invertible. Then to every point g we can associate the set map:

$$\tilde{g} := t \circ (s|_{W_g})^{-1} : s(W_g) \to t(W_g)$$

which is a biholomorphism between two open neighborhoods of $\tilde{x}$. Then we can define the set map:

$$f_{\tilde{x}} : R_{\tilde{x}} \to HolGerms_{\tilde{x}}(U) \tag{2.7}$$

(where $HolGerms_{\tilde{x}}(U)$ is the group of germs of holomorphic maps defined on an open neighborhood of $\tilde{x}$ in U and which fix $\tilde{x}$), that to every $g \in R_{\tilde{x}}$ associates the germ of the function $\tilde{g}$ at the point $\tilde{x}$. Then we say that the groupoid $R \overset{s}{\underset{t}{\rightrightarrows}} U$ is *effective* (or *reduced*) if $f_{\tilde{x}}$ is injective for every $\tilde{x}$ in U. This notion will correspond to the notion of reduced orbifolds via the 2-functor F that we'll define in Section 3.

We have this useful result:

Lemma 2.16. *The following identities hold on any open connected neighborhood of* $s(g)$ *where both the left hand side and the right hand side are defined*:

(a) *for every pair of points* $(g,h) \in R_t \times_s R$ *if we call* $k := m(g,h)$*, we have that* $\tilde{h} \circ \tilde{g} = \tilde{k}$;
(b) *for every* $g \in R$ *we have that* $\widetilde{i(g)} = \tilde{g}^{-1}$.

Corollary 2.17. *The set map of* (2.7) *is a group homomorphism.*

Now let us recall the following result (stated in the smooth case, but also true in the holomorphic case).

Proposition 2.18. *([18], §2.1) The data of Lie groupoids, morphisms and natural transformations between them form a* 2*-category, known as* (**LieGpd**).

Definition 2.19. Let us define the following 2-category, that we will denote with (**Grp**):

- the objects are all the *proper, étale and effective groupoid objects* in the category (**Manifolds**) of complex manifolds and holomorphic maps between them;

- the morphisms between these objects are *all* the morphisms between them as groupoid objects in (**Manifolds**);
- the 2-morphisms are *all* the natural transformations between the morphisms of the previous point.

Since every étale groupoid object is in particular a Lie groupoid, we are just considering a full sub-2-category of (**LieGpd**), hence we have for free that:

Proposition 2.20. (**Grp**) *is actually a* 2*-category.*

3. From orbifolds to groupoids

Our aim is to describe a 2-functor F from (**Pre-Orb**) to (**Grp**). This will be done on the level of objects by adapting to the complex case a construction due to D. Pronk (see [16], §4.4) in the smooth case: we associate to every atlas $\mathcal{U}$ a groupoid object in (**Sets**), then we prove that it is an object of (**Grp**). The original part of this section consists in the extension of this construction to the level of morphisms and 2-morphisms. The last part of this section will be devoted to prove that actually this gives rise to a 2-functor F from (**Pre-Orb**) to (**Grp**).

3.1. Objects

Let us fix any orbifold atlas $\mathcal{U}=\{(\widetilde{U}_i,G_i,\pi_i)\}_{i\in I}$ of dimension n on a paracompact and second countable Hausdorff topological space X; we want to associate to it a groupoid object $R \underset{t}{\overset{s}{\rightrightarrows}} U$ which "encodes" the information about the underlying topological space X and the atlas $\mathcal{U}$. First of all, we define:

$$U:=\coprod_{i\in I}\widetilde{U}_i$$

with the topology of the disjoint union. Since all the $\widetilde{U}_i$'s are open subsets of $\mathbb{C}^n$ and U is their disjoint union, then U is a Hausdorff paracompact complex manifold of dimension n. The points of this manifold will be always denoted as $(\tilde{x}_i,\widetilde{U}_i)$ if $\tilde{x}_i\in\widetilde{U}_i\subseteq U$. In the following constructions we will tacitly assume that if we take a generic point $\tilde{x}_i$, then this point belongs to $\widetilde{U}_i$.

Now the idea is that whenever we have U defined in this way, we would like to recover both the underlying topological space X (up to homeomorphism, see Proposition 4.5 below) and the atlas $\mathcal{U}$; how to do this must be encoded in R, that we are going to define. First of all, we define a manifold

$$\hat{R}:=\coprod\widetilde{U}_k^{ij}$$

where $\widetilde{U}_k^{ij}:=\widetilde{U}_k^{\lambda_{ki},\lambda_{kj}}$ denotes a copy of $\widetilde{U}_k$ indexed by a pair of embeddings $\lambda_{ki},\lambda_{kj}$, and where the disjoint union is taken *over all triples of uniformizing systems of the form* :

$$(\widetilde{U}_i,G_i,\pi_i)\xleftarrow{\lambda_{ki}}(\widetilde{U}_k,G_k,\pi_k)\xrightarrow{\lambda_{kj}}(\widetilde{U}_j,G_j,\pi_j). \tag{3.1}$$

In other words, the disjoint union is taken over all the charts $(\widetilde{U}_k,G_k,\pi_k)\in\mathcal{U}$ and over all the possible embeddings of them into any pair of uniformizing systems as in (3.1); we will write $(\lambda_{ki},\tilde{x}_k,\lambda_{kj})$ to denote the point $\tilde{x}_k\in\widetilde{U}_k$ considered as in the set $\widetilde{U}_k^{ij}\subseteq\hat{R}$. Now we could easily define 4 of the 5 morphisms of a groupoid as follows:

$$\hat{s}(\lambda_{ki},\tilde{x}_k,\lambda_{kj}):=\big(\lambda_{ki}(\tilde{x}_k),\widetilde{U}_i\big),\qquad \hat{t}(\lambda_{ki},\tilde{x}_k,\lambda_{kj}):=\big(\lambda_{kj}(\tilde{x}_k),\widetilde{U}_j\big),$$
$$\hat{e}(\tilde{x}_i,\widetilde{U}_i):=(1_{\widetilde{U}_i},\tilde{x}_i,1_{\widetilde{U}_i}),\qquad \hat{i}(\lambda_{ki},\tilde{x}_k,\lambda_{kj}):=(\lambda_{kj},\tilde{x}_k,\lambda_{ki}).$$

$\hat{R}$ is *almost* the manifold we want to use; the only problem is that we can't have a well-defined notion of multiplication m on it, so we have to define R as a quotient of $\hat{R}$ by a locally trivial relation of equivalence as follows. The necessity of such a relation will be clear in the proof of Lemma 3.7.

Definition 3.1. ([16], §4.4) Two points $(\lambda_{ki},\tilde{x}_k,\lambda_{kj})$ and $(\lambda_{li},\tilde{x}_l,\lambda_{lj})$ in $\hat{R}$ are called *equivalent* and we write:

$$(\lambda_{ki},\tilde{x}_k,\lambda_{kj})\sim(\lambda_{li},\tilde{x}_l,\lambda_{lj}) \tag{3.2}$$

if there exist a uniformizing system $(\widetilde{U}_m,G_m,\pi_m)\in\mathcal{U}$, a point $\tilde{x}_m\in\widetilde{U}_m$ and two embeddings:

$$(\widetilde{U}_k,G_k,\pi_k)\xleftarrow{\lambda_{mk}}(\widetilde{U}_m,G_m,\pi_m)\xrightarrow{\lambda_{ml}}(\widetilde{U}_l,G_l,\pi_l)$$

such that the following diagrams are commutative:

$$
\begin{tikzcd}
& \widetilde{U}_k \arrow[dl, "\lambda_{ki}"'] \arrow[dr, "\lambda_{kj}"] & \\
\widetilde{U}_i \arrow[r, phantom, "\curvearrowright"] & \widetilde{U}_m \arrow[u, "\lambda_{mk}"'] \arrow[d, "\lambda_{ml}"] \arrow[r, phantom, "\curvearrowright"] & \widetilde{U}_j \\
& \widetilde{U}_l \arrow[ul, "\lambda_{li}"] \arrow[ur, "\lambda_{lj}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \tilde{x}_k \arrow[dl, "\lambda_{ki}"'] \arrow[dr, "\lambda_{kj}"] & \\
\tilde{x}_i \arrow[r, phantom, "\curvearrowright"] & \tilde{x}_m \arrow[u, "\lambda_{mk}"'] \arrow[d, "\lambda_{ml}"] \arrow[r, phantom, "\curvearrowright"] & \tilde{x}_j. \\
& \tilde{x}_l \arrow[ul, "\lambda_{li}"] \arrow[ur, "\lambda_{lj}"'] &
\end{tikzcd}
\tag{3.3}
$$

Remark 3.2. In order to simplify the notation, *here and from now on we omit the groups G_i's and the maps π_i's* in every diagram; in other words from now on every map λ_{ij} will be an embedding between uniformizing systems even if we write only its source and target as open sets of $\mathbb{C}^n$ and not as uniformizing systems.

Now our aim is to prove that (3.2) *is an equivalence relation on* R'. In order to do that, let us state the following lemma, whose original proof in the differentiable case is due to D. Pronk (see [16], Lemma 4.4.1).

Lemma 3.3. *Let us fix an atlas $\mathcal{U}$, a pair of embeddings $\lambda_{nl} : (\widetilde{U}_n, G_n, \pi_n) \to (\widetilde{U}_l, G_l, \pi_l)$, $\lambda_{pl} : (\widetilde{U}_p, G_p, \pi_p) \to (\widetilde{U}_l, G_l, \pi_l)$ and a pair of points $\tilde{x}_n \in \widetilde{U}_n$, $\tilde{x}_p \in \widetilde{U}_p$ such that $\lambda_{nl}(\tilde{x}_n) = \lambda_{pl}(\tilde{x}_p) =: \tilde{x}_l$. Then there exist a uniformizing system $(\widetilde{U}_q, G_q, \pi_q) \in \mathcal{U}$, a pair of embeddings $\lambda_{qn}, \lambda_{qp}$ and a point $\tilde{x}_q \in \widetilde{U}_q$ such that the following diagrams are commutative*:

$$
\begin{tikzcd}
& \widetilde{U}_q \arrow[dl, dashed, "\lambda_{qn}"'] \arrow[dr, dashed, "\lambda_{qp}"] & \\
\widetilde{U}_n \arrow[dr, "\lambda_{nl}"'] \arrow[rr, phantom, "\curvearrowright"] & & \widetilde{U}_p \arrow[dl, "\lambda_{pl}"] \\
& \widetilde{U}_l &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \tilde{x}_q \arrow[dl, dashed, "\lambda_{qn}"'] \arrow[dr, dashed, "\lambda_{qp}"] & \\
\tilde{x}_n \arrow[dr, "\lambda_{nl}"'] \arrow[rr, phantom, "\curvearrowright"] & & \tilde{x}_p. \arrow[dl, "\lambda_{pl}"] \\
& \tilde{x}_l &
\end{tikzcd}
\tag{3.4}
$$

As a direct application of this lemma one can prove that $\sim$ is transitive (in the differentiable case, this is [16], Lemma 4.4.2); since $\sim$ is clearly reflexive and symmetric we get:

Lemma 3.4. $\sim$ *is an equivalence relation on* $\hat{R}$.

Definition 3.5. For every orbifold atlas $\mathcal{U}$ over X, we define the set $R := \hat{R}/\sim$ and we call $q : \hat{R} \to R$ the quotient map; we will denote the class of any point $(\lambda_{ki}, \tilde{x}_k, \lambda_{kj})$ with $[\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]$.

Remark 3.6. The equivalence relation we have just described is the same as the relation $\sim$ defined in [14] as the relation *generated* by considering equivalent $(\lambda_{ki}, \tilde{x}_k, \lambda_{kj})$ and $(\lambda_{li}, \tilde{x}_l, \lambda_{lj})$ whenever there exists an embedding λ_{lk} such that the following diagrams are commutative:

$$
\begin{tikzcd}
& \widetilde{U}_l \arrow[dl, "\lambda_{li}"'] \arrow[d, "\lambda_{lk}"] \arrow[dr, "\lambda_{lj}"] \arrow[dl, phantom, "\curvearrowright", shift left=3ex] \arrow[dr, phantom, "\curvearrowright", shift right=3ex] & \\
\widetilde{U}_i & \widetilde{U}_k \arrow[l, "\lambda_{ki}"] \arrow[r, "\lambda_{kj}"'] & \widetilde{U}_j
\end{tikzcd}
\qquad
\begin{tikzcd}
& \tilde{x}_l \arrow[dl, "\lambda_{li}"'] \arrow[d, "\lambda_{lk}"] \arrow[dr, "\lambda_{lj}"] \arrow[dl, phantom, "\curvearrowright", shift left=3ex] \arrow[dr, phantom, "\curvearrowright", shift right=3ex] & \\
\tilde{x}_i & \tilde{x}_k \arrow[l, "\lambda_{ki}"] \arrow[r, "\lambda_{kj}"'] & \tilde{x}_j.
\end{tikzcd}
\tag{3.5}
$$

Hence from now on we will use without distinction the first and the second equivalence; in the following pages we will often have to define set maps on R using representatives of the equivalence classes; in order to prove that they are well defined it will be sufficient to use different representatives related by a diagram of the form (3.5).

Now we recall that our first aim is to make the pair (R, U) into a groupoid object in (**Sets**), so for the moment we don't care about the topology on R and we consider it just as a set. The maps s, t, i, e are just induced by the maps

$\hat{s}, \hat{t}, \hat{\imath}, \hat{e}$ respectively; the fact that the first 3 maps does not depend on the representatives chosen is just a straightforward application of Remark 3.6. Now we want to define the "multiplication" on R, so let us consider any pair of "composable arrows" $[\lambda_{ih}, \tilde{x}_i, \lambda_{ij}]$ and $[\lambda_{kj}, \tilde{x}_k, \lambda_{kl}]$ in the set-theoretical fiber product $R_t \times_s R$. In other words, let us assume that $t([\lambda_{ih}, \tilde{x}_i, \lambda_{ij}]) = s([\lambda_{kj}, \tilde{x}_k, \lambda_{kl}])$ i.e. $\lambda_{ij}(\tilde{x}_i) = \lambda_{kj}(\tilde{x}_k)$; equivalently, we have a diagram of uniformizing systems and marked points as follows:

$$\begin{tikzcd}
 & \tilde{x}_i \arrow[d, phantom, "\Cap" description] & \lambda_{ij}(\tilde{x}_i) = \lambda_{kj}(\tilde{x}_k) \arrow[d, phantom, "\Cap" description] & \tilde{x}_k \arrow[d, phantom, "\Cap" description] & \\
\widetilde{U}_h & \widetilde{U}_i \arrow[l, "\lambda_{ih}"'] \arrow[r, "\lambda_{ij}"] & \widetilde{U}_j & \widetilde{U}_k \arrow[l, "\lambda_{kj}"'] \arrow[r, "\lambda_{kl}"] & \widetilde{U}_l.
\end{tikzcd} \tag{3.6}$$

Since $\lambda_{ij}(\tilde{x}_i) = \lambda_{kj}(\tilde{x}_k)$, we can apply Lemma 3.3, so we get that there exist a uniformizing system $(\widetilde{U}_f, G_f, \pi_f)$, a point $\tilde{x}_f \in \widetilde{U}_f$ and embeddings $\lambda_{fi}, \lambda_{fk}$ such that we can complete (3.6) to a commutative diagram (also on the level of marked points):

$$\begin{tikzcd}
 & \tilde{x}_i \arrow[d, phantom, "\Cap" description] & \lambda_{ij}(\tilde{x}_i) = \lambda_{kj}(\tilde{x}_k) \arrow[d, phantom, "\Cap" description] & \tilde{x}_k \arrow[d, phantom, "\Cap" description] & \\
\widetilde{U}_h & \widetilde{U}_i \arrow[l, "\lambda_{ih}"'] \arrow[r, "\lambda_{ij}"] & \widetilde{U}_j \arrow[d, phantom, "\curvearrowright" description] & \widetilde{U}_k \arrow[l, "\lambda_{kj}"'] \arrow[r, "\lambda_{kl}"] & \widetilde{U}_l. \\
 & & \widetilde{U}_f \arrow[ul, "\lambda_{fi}"] \arrow[ur, "\lambda_{fk}"'] & & \\
 & & \tilde{x}_f \arrow[u, phantom, "\Cup" description] & &
\end{tikzcd} \tag{3.7}$$

Hence we give the following definition:

$$m\big([\lambda_{ih}, \tilde{x}_i, \lambda_{ij}], [\lambda_{kj}, \tilde{x}_k, \lambda_{kl}]\big) := [\lambda_{ih} \circ \lambda_{fi}, \tilde{x}_f, \lambda_{kl} \circ \lambda_{fk}].$$

Lemma 3.7. *The map m is well defined (see Appendix* A *for the proof).*

Proposition 3.8. $(R \overset{s}{\underset{t}{\rightrightarrows}} U)$ *is a groupoid object in* (**Sets**).

Proof. We have to prove that all the axioms for a groupoid object given in Definition 2.1 are satisfied; the only non-trivial one is the third one, so let us fix any triple of composable arrows:

$$\big([\lambda_{ih}, \tilde{x}_i, \lambda_{ij}], [\lambda_{kj}, \tilde{x}_k, \lambda_{kl}], [\lambda_{ml}, \tilde{x}_m, \lambda_{mn}]\big) \in R\,{}_t\times_s R\,{}_t\times_s R$$

i.e. $\lambda_{ij}(\tilde{x}_i) = \lambda_{kj}(\tilde{x}_k)$ and $\lambda_{kl}(\tilde{x}_k) = \lambda_{ml}(\tilde{x}_m)$. If we apply Lemma 3.3 twice, we get a commutative diagram (of uniformizing systems and marked points) as follows:

$$\begin{tikzcd}
 & \tilde{x}_i \arrow[d, phantom, "\Cap" description] & & \tilde{x}_k \arrow[d, phantom, "\Cap" description] & & \tilde{x}_m \arrow[d, phantom, "\Cap" description] & \\
\widetilde{U}_h & \widetilde{U}_i \arrow[l, "\lambda_{ih}"'] \arrow[r, "\lambda_{ij}"] & \widetilde{U}_j \arrow[d, phantom, "\curvearrowright" description] & \widetilde{U}_k \arrow[l, "\lambda_{kj}"'] \arrow[r, "\lambda_{kl}"] & \widetilde{U}_l \arrow[d, phantom, "\curvearrowright" description] & \widetilde{U}_m \arrow[l, "\lambda_{ml}"'] \arrow[r, "\lambda_{mn}"] & \widetilde{U}_n. \\
 & & \widetilde{U}_f \arrow[ul, "\lambda_{fi}"] \arrow[ur, "\lambda_{fk}"'] & & \widetilde{U}_s \arrow[ul, "\lambda_{sk}"] \arrow[ur, "\lambda_{sm}"'] & & \\
 & & \tilde{x}_f \arrow[u, phantom, "\Cup" description] & & \tilde{x}_s \arrow[u, phantom, "\Cup" description] & &
\end{tikzcd}$$

If we consider the central part of it, we can apply again Lemma 3.3 in order to get a commutative diagram of the form:

$$
\begin{tikzcd}
 & & \tilde{x}_i \arrow[d, phantom, sloped, "\in"] & & & & \tilde{x}_k \arrow[d, phantom, sloped, "\in"] & & & & \tilde{x}_m \arrow[d, phantom, sloped, "\in"] & & \\
\widetilde{U}_h & & \widetilde{U}_i \arrow[ll, "\lambda_{ih}"'] \arrow[rr, "\lambda_{ij}"] \arrow[rrrr, phantom, "\circlearrowright"'] & & \widetilde{U}_j & & \widetilde{U}_k \arrow[ll, "\lambda_{kj}"'] \arrow[rr, "\lambda_{kl}"] \arrow[rrrr, phantom, "\circlearrowright"'] & & \widetilde{U}_l & & \widetilde{U}_m \arrow[ll, "\lambda_{ml}"'] \arrow[rr, "\lambda_{mn}"] & & \widetilde{U}_n . \\
 & & & & \widetilde{U}_f \arrow[ull, "\lambda_{fi}"] \arrow[urr, "\lambda_{fk}"'] \arrow[rrrr, phantom, "\circlearrowright"] & & & & \widetilde{U}_s \arrow[ull, "\lambda_{sk}"] \arrow[urr, "\lambda_{sm}"'] & & & & \\
 & & & \tilde{x}_f \arrow[ur, phantom, sloped, "\in"] & & & & & & \tilde{x}_s \arrow[ul, phantom, sloped, "\ni"] & & & \\
 & & & & & & \widetilde{U}_r \arrow[uull, "\lambda_{rf}"] \arrow[uurr, "\lambda_{rs}"'] & & & & & & \\
 & & & & & & \tilde{x}_r \arrow[u, phantom, sloped, "\in"] & & & & & &
\end{tikzcd}
\tag{3.8}
$$

Now:

$$
\begin{aligned}
m\big([\lambda_{kj}, \tilde{x}_k, \lambda_{kl}], [\lambda_{ml}, \tilde{x}_m, \lambda_{mn}]\big) &= [\lambda_{kj} \circ \lambda_{sk} \circ \lambda_{rs}, \tilde{x}_r, \lambda_{mn} \circ \lambda_{sm} \circ \lambda_{rs}] \\
&= [\lambda_{kj} \circ \lambda_{fk} \circ \lambda_{rf}, \tilde{x}_r, \lambda_{mn} \circ \lambda_{sm} \circ \lambda_{rs}];
\end{aligned}
$$

hence using the commutative diagram:

$$
\begin{tikzcd}
 & & \tilde{x}_i \arrow[d, phantom, sloped, "\in"] & & \lambda_{ij}(\tilde{x}_i) = \lambda_{kj}(\tilde{x}_k) \arrow[d, phantom, sloped, "\in"] & & \tilde{x}_r \arrow[d, phantom, sloped, "\in"] & & \\
\widetilde{U}_h & & \widetilde{U}_i \arrow[ll, "\lambda_{ih}"'] \arrow[rr, "\lambda_{ij}"] & & \widetilde{U}_j & & \widetilde{U}_r \arrow[ll, "\lambda_{kj} \circ \lambda_{fk} \circ \lambda_{rf}"'] \arrow[rr, "\lambda_{mn} \circ \lambda_{sm} \circ \lambda_{rs}"] & & \widetilde{U}_n \\
 & & & & \widetilde{U}_r \arrow[ull, "\lambda_{fi} \circ \lambda_{rf}"] \arrow[urr, "1_{\widetilde{U}_r}"'] \arrow[u, phantom, "\circlearrowright"] & & & & \\
 & & & & \tilde{x}_r \arrow[u, phantom, sloped, "\in"] & & & &
\end{tikzcd}
$$

we can compute:

$$
\begin{aligned}
& m\big([\lambda_{ih}, \tilde{x}_i, \lambda_{ij}], m\big([\lambda_{kj}, \tilde{x}_k, \lambda_{kl}], [\lambda_{ml}, \tilde{x}_m, \lambda_{mn}]\big)\big) \\
&\quad = m\big([\lambda_{ih}, \tilde{x}_i, \lambda_{ij}], [\lambda_{kj} \circ \lambda_{fk} \circ \lambda_{rf}, \tilde{x}_r, \lambda_{mn} \circ \lambda_{sm} \circ \lambda_{rs}]\big) \\
&\quad = [\lambda_{ih} \circ \lambda_{fi} \circ \lambda_{rf}, \tilde{x}_r, \lambda_{mn} \circ \lambda_{sm} \circ \lambda_{rs}].
\end{aligned}
$$

Since diagram (3.8) is symmetric, we obtain the same result also if we first compute the multiplication of the first two arrows, and then we multiply them with the third one. In other words, we have proved that $m \circ (1_R \times m) = m \circ (m \times 1_R)$. □

Now we want to describe a structure of complex manifold on R.

Lemma 3.9. *The equivalence relation $\sim$ is the trivial one whenever we restrict to any open set of $\hat{R}$ of the form $\widetilde{U}_k^{ij}$.*

This is just a direct consequence of the definition of $\sim$ restricted to any pair of points of the form $(\lambda_{ki}, \tilde{x}_k, \lambda_{kj})$ and $(\lambda_{ki}, \tilde{x}'_k, \lambda_{kj})$.

Proposition 3.10. *If we give to $R = \hat{R}/\sim$ the quotient topology, then we get a natural structure of complex manifold.*

Proof. Let us consider any open set of $\hat{R}$ of the form $A := \widetilde{U}_k^{ij}$ and let us denote with A^{sat} the saturated of A in $\hat{R}$ with respect to $\sim$. *We claim that also A^{sat} is open in $\hat{R}$.* Indeed, let us consider any point in A^{sat}, i.e. a point which is equivalent to a point $(\lambda_{ki}, \tilde{x}_k, \lambda_{kj})$ of $\widetilde{U}_k^{ij}$. By definition of $\sim$, this must be necessarily of the form $(\lambda_{li}, \tilde{x}_l, \lambda_{lj})$; moreover, there must exist a uniformizing system $(\widetilde{U}_m, G_m, \pi_m)$, a point $\tilde{x}_m$ and embeddings $\lambda_{mk}, \lambda_{ml}$ as in (3.3).

Now let us consider the set $\widetilde{B} := \lambda_{ml}(\widetilde{U}_m) \subseteq \widetilde{U}_l$, which is an open neighborhood of $\tilde{x}_l$ (because λ_{ml} is an embedding between open sets of $\mathbb{C}^n$, where n is the dimension of the orbifold atlas $\mathcal{U}$). If we fix any other point $\tilde{x}'_m$ in $\widetilde{U}_m$ we get a diagram similar to the second one of (3.3), so the set $\widetilde{B}$ (considered as an open subset of $\widetilde{U}_l^{ij}$, hence also as an open set of $\hat{R}$) contains only points equivalent to points of A, so is completely contained in A^{sat}; hence A^{sat} is open in $\hat{R}$.

So if $q : \hat{R} \to R$ is the quotient map, the set $q(A^{\mathrm{sat}})$ is open in R by definition of quotient topology; moreover, by definition of saturated, it coincides with $q(A)$. Since this holds for every choice of $A = \widetilde{U}_k^{ij}$ we get that the family $\{\widetilde{W}_k^{ij} := q(\widetilde{U}_k^{ij}) = q(\widetilde{U}_k^{ij\ \mathrm{sat}})\}_{\widetilde{U}_k^{ij} \subseteq \hat{R}}$ is an open covering of R (in the quotient topology). Then our aim is to construct from it a complex manifold atlas on R. If we use the previous lemma, we get that $\sim$ is the trivial equivalence relation on every set $\widetilde{U}_k^{ij}$, so $q(\widetilde{U}_k^{ij})$ is homeomorphic to $\widetilde{U}_k^{ij}$ via q (which is invertible if we restrict to this set). Moreover, we recall that by construction $\widetilde{U}_k^{ij}$ is just a copy of $\widetilde{U}_k$, so the set map ϕ_k^{ij} defined from $\widetilde{W}_k^{ij}$ to $\widetilde{U}_k$ as:

$$\phi_k^{ij}\big([\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]\big) := \tilde{x}_k$$

is a homeomorphism (with codomain an open subset of $\mathbb{C}^n$). So it makes sense to consider the family of charts $\mathcal{F} := \{(\widetilde{W}_k^{ij}, \phi_k^{ij})\}_{\widetilde{U}_k^{ij} \subseteq \hat{R}}$. Since the domains of these charts cover all R, it remains only to prove the compatibility condition on the intersection of any pair of charts; so let us fix any pair of domains $\widetilde{W}_k^{ij}$ and $\widetilde{W}_l^{i'j'}$ such that $\widetilde{W}_k^{ij} \cap \widetilde{W}_l^{i'j'}$ is non-empty and let us fix any point $P = [\lambda_{ki}, \tilde{x}_k, \lambda_{kj}] = [\lambda_{li'}, \tilde{x}_l, \lambda_{lj'}]$ in the intersection. By definition of $\sim$, we get that necessarily $i' = i$ and $j' = j$; moreover, there exist a uniformizing system $(\widetilde{U}_m, G_m, \pi_m)$, a point $\tilde{x}_m \in \widetilde{U}_m$ and a pair of embeddings $\lambda_{mk}, \lambda_{ml}$ as in (3.3). Now the images of the point P via the coordinate functions ϕ_k^{ij} and ϕ_l^{ij} are respectively:

$$\phi_k^{ij}\big([\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]\big) = \tilde{x}_k = \lambda_{mk}(\tilde{x}_m) \quad \text{and} \quad \phi_l^{ij}\big([\lambda_{li}, \tilde{x}_l, \lambda_{lj}]\big) = \tilde{x}_l = \lambda_{ml}(\tilde{x}_m).$$

So if we call ϕ the transition map:

$$\phi = \phi_l^{ij} \circ \big(\phi_k^{ij}\big)^{-1} : \phi_k^{ij}\big(\widetilde{W}_k^{ij} \cap \widetilde{W}_l^{ij}\big) \to \phi_l^{ij}\big(\widetilde{W}_k^{ij} \cap \widetilde{W}_l^{ij}\big),$$

we get that $\phi(\tilde{x}_k) = \tilde{x}_l = \lambda_{ml}(\tilde{x}_m) = \lambda_{ml} \circ (\lambda_{mk}|_{\lambda_{mk}(\widetilde{U}_m)})^{-1}(\tilde{x}_k)$. As before, using diagram (3.3) we get that this is the expression of ϕ not only at the point $\tilde{x}_k$, but also in an open neighborhood of it (not necessarily coinciding with all the domain of ϕ). Hence we have proved that the transition map ϕ locally coincides with a holomorphic function. So every transition function is holomorphic, hence we have proved that the family $\mathcal{F}$ is a complex manifold atlas for R. $\square$

Lemma 3.11. *$R \overset{s}{\underset{t}{\rightrightarrows}} U$ is an étale groupoid object in* (**Manifolds**).

Proof. We have already proved that $R \overset{s}{\underset{t}{\rightrightarrows}} U$ is a groupoid object in (**Sets**), and that both R and U are complex manifolds. Hence we have only to prove the additional properties about the five structure maps. In particular, we have to prove that s and t are both étale (hence, in particular, holomorphic) and that m, i and e are holomorphic.

Let us prove that s is étale (the proof for t is analogous); since the property of being étale is a local one, we can check it by restricting to the domains of suitable charts in source and target. So let us fix any point $[\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]$ in R and the chart $(\widetilde{W}_k^{ij}, \phi_k^{ij})$ around it. We recall that:

$$s\big([\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]\big) = \lambda_{ki}(\tilde{x}_k) \in \widetilde{U}_i \subseteq U$$

where $\widetilde{U}_i$ means a copy of $\widetilde{U}_i$ in the manifold U; so a chart around this point is just $(\widetilde{U}_i, id)$. Hence the map s can be expressed in coordinates as:

$$\tilde{s} := id \circ s \circ \big(\phi_k^{ij}\big)^{-1} : \widetilde{U}_k \to \widetilde{U}_i$$

which coincides with the holomorphic embedding λ_{ki}. So s is a biholomorphism if restricted to $\widetilde{W}_k^{ij}$ in domain and to $\lambda_{ki}(\widetilde{U}_k)$ in codomain. Hence we have proved that s is étale.

In order to prove that $m : R_t \times_s R \to R$ is holomorphic, let us fix any point:

$$\big(P, P'\big) := \big([\lambda_{ih}, \tilde{x}_i, \lambda_{ij}], [\lambda_{kj}, \tilde{x}_k, \lambda_{kl}]\big) \in R_t \times_s R$$

and a completion of the form (3.7). Then we can write:

$$\big(P, P'\big) = \big([\lambda_{fh}, \tilde{x}_f, \lambda_{fj}], [\lambda_{fj}, \tilde{x}_f, \lambda_{fl}]\big)$$

(where we define $\lambda_{fh} := \lambda_{ih} \circ \lambda_{fi}$ and analogously for the other 3 embeddings). Now let us define a set map $\delta : \widetilde{U}_f \to R \times R$ as:

$$\delta(\tilde{y}_f) := \big([\lambda_{fh}, \tilde{y}_f, \lambda_{fj}], [\lambda_{fj}, \tilde{y}_f, \lambda_{fl}]\big).$$

This map is clearly holomorphic because $R \times R$ has the product topology and by combining δ with the first and second projection we get exactly inverses of holomorphic coordinates functions on R (see the explicit description in Proposition 3.10).

Moreover, one proves easily that δ has target in $R_t \times_s R$, which is a complex submanifold of $R \times R$ (see Proposition 2.10), hence δ is holomorphic from $\widetilde{U}_f$ to $R_t \times_s R$. In addition, there exists an obvious local inverse of δ, again holomorphic, hence δ is a biholomorphism if restricted in target to an open neighborhood of (P, P'). Now using a diagram similar to (3.7) we get that $m \circ \delta(\tilde{y}_f) = [\lambda_{fh}, \tilde{y}_f, \lambda_{fl}]$; since a chart around $m(P, P')$ is just $(\widetilde{W}^{hl}_f, \phi^{hl}_f)$, we get that in order to check whether m is holomorphic or not around (P, P'), it suffices to check if $\phi^{hl}_f \circ m \circ \delta$ is holomorphic near $\tilde{x}_f$. Now this map just coincides with the identity on the whole $\widetilde{U}_f$, hence m is holomorphic around (P, P'); since this holds for every point of $R_t \times_s R$, we are done.

Analogous arguments prove that both i and e are holomorphic maps. $\square$

Lemma 3.12. *Suppose we have fixed 2 points $P = [\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]$ and $Q = [\lambda_{li}, \tilde{x}_l, \lambda_{lj}]$ of R with $s(P) = s(Q)$ and $t(P) = t(Q)$. Then there exists a unique $g \in G_j$ such that P can be written as*:

$$[\lambda_{ki}, \tilde{x}_k, \lambda_{kj}] = [\lambda_{li}, \tilde{x}_l, g \circ \lambda_{lj}]. \tag{3.9}$$

Moreover, such a g belongs to the stabilizer of $\lambda_{lj}(\tilde{x}_l)$ in $\widetilde{U}_j$, so by Lemmas 1.14 *and* 1.12 *there exists a unique $g' \in (G_l)_{\tilde{x}_l}$ such that*:

$$[\lambda_{ki}, \tilde{x}_k, \lambda_{kj}] = \left[\lambda_{li}, \tilde{x}_l, \lambda_{lj} \circ g'\right].$$

Proof. The hypothesis implies that $\lambda_{ki}(\tilde{x}_k) = \lambda_{li}(\tilde{x}_l)$ and $\lambda_{kj}(\tilde{x}_k) = \lambda_{lj}(\tilde{x}_l)$. Using the first relation, we can apply Lemma 3.3 to the pair of embeddings $\lambda_{ki}, \lambda_{li}$, so we get a pair of diagrams of the form (3.3), which a priori are both commutative only in the left part. Then one can apply Lemma 1.12 to the right part, so we get a unique $g \in G_j$ such that we have commutative diagrams of the form:

$$\begin{tikzcd}
& \widetilde{U}_k \arrow[dl, "\lambda_{ki}"'] \arrow[dr, "\lambda_{kj}"] & \\
\widetilde{U}_i \arrow[r, phantom, "\curvearrowright"] & \widetilde{U}_m \arrow[u, "\lambda_{mk}"'] \arrow[d, "\lambda_{ml}"] \arrow[r, phantom, "\curvearrowright"] & \widetilde{U}_j \\
& \widetilde{U}_l \arrow[ul, "\lambda_{li}"] \arrow[ur, "g\circ\lambda_{lj}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \tilde{x}_k \arrow[dl, "\lambda_{ki}"'] \arrow[dr, "\lambda_{kj}"] & \\
\tilde{x}_i \arrow[r, phantom, "\curvearrowright"] & \tilde{x}_m \arrow[u, "\lambda_{mk}"'] \arrow[d, "\lambda_{ml}"] \arrow[r, phantom, "\curvearrowright"] & \tilde{x}_j. \\
& \tilde{x}_l \arrow[ul, "\lambda_{li}"] \arrow[ur, "g\circ\lambda_{lj}"'] &
\end{tikzcd} \tag{3.10}$$

Hence, by definition of $\sim$ on $\hat{R}$, we get that (3.9) is satisfied. Moreover, it is simple to prove that such a g is also unique using again Lemma 1.12. $\square$

Lemma 3.13. *The étale groupoid object $R \underset{t}{\overset{s}{\rightrightarrows}} U$ is effective.*

Proof. Let us fix any point $(\tilde{x}_k, \widetilde{U}_k) \in U$; by applying the previous lemma we get that the set of points P of R such that $s(P) = t(P) = (\tilde{x}_k, \widetilde{U}_k)$ is in natural bijection with the stabilizer $(G_k)_{\tilde{x}_k}$; in particular, the bijection is given by $g \mapsto [1_{\widetilde{U}_k}, \tilde{x}_k, g]$. Now for every $P = [1_{\widetilde{U}_k}, \tilde{x}_k, g]$, if we restrict to any sufficiently small open neighborhood W_g around P, we get the induced map $t \circ (s|_{W_g})^{-1} = g$ around $\tilde{x}_k$. Since the orbifold atlas $\mathcal{U}$ is reduced by hypothesis, the group G_k acts effectively on $\widetilde{U}_k$, hence the set map $f_{(\tilde{x}_k, \widetilde{U}_k)}$ (see Definition 2.15) is injective. Since this holds for every point of U, we have proved that $R \underset{t}{\overset{s}{\rightrightarrows}} U$ is effective. $\square$

Lemma 3.14. *The relative diagonal $(s, t) : R \to U \times U$ is proper.*

Proof. (Adapted from [16], Proposition 4.4.8 and Corollary 4.4.9.) Let us fix any point $(u, u') = ((\tilde{x}_i, \widetilde{U}_i), (\tilde{x}_j, \widetilde{U}_j)) \in U \times U$ and let us distinguish between 2 cases: if $\pi_i(\tilde{x}_i) \neq \pi_j(\tilde{x}_j)$, then we can use the fact that X is Hausdorff (by definition of orbifold) and we get that there exist two open disjoint neighborhoods D_i and D_j of $\pi_i(\tilde{x}_i)$ and $\pi_j(\tilde{x}_j)$. If we call $\widetilde{D}_i := \pi_i^{-1}(D_i) \subseteq \widetilde{U}_i$ and $\widetilde{D}_j := \pi_j^{-1}(D_j) \subseteq \widetilde{U}_j$, we get that $\widetilde{D}_i \times \widetilde{D}_j$ is an open neighborhood of (u, u') and its preimage via (s, t) is empty.

Now let us consider the case when $\pi_i(\tilde{x}_i) = \pi_j(\tilde{x}_j)$; in this case we can use property (ii) of orbifold atlases and Remark 1.17 in order to find a uniformizing system $(\widetilde{U}_k, G_k, \pi_k) \in \mathcal{U}$, a point $\tilde{x}_k \in \widetilde{U}_k$ and embeddings $\lambda_{ki}, \lambda_{kj}$ such that $\lambda_{ki}(\tilde{x}_k) = \tilde{x}_i$ and $\lambda_{kj}(\tilde{x}_k) = \tilde{x}_j$. Then let us consider the open sets $\widetilde{W}_i := \lambda_{ki}(\widetilde{U}_k) \subseteq \widetilde{U}_i$, $\widetilde{W}_j := \lambda_{kj}(\widetilde{U}_k) \subseteq \widetilde{U}_j$ and the set

$\widetilde{W}_i \times \widetilde{W}_j$, which is an open neighborhood of (u, u') in $U \times U$. Now let us fix any point:

$$P = [\lambda_{li}, \tilde{y}_l, \lambda_{lj}] \in (s,t)^{-1}(\widetilde{W}_i \times \widetilde{W}_j)$$

and let us define $\tilde{y}_k := \lambda_{ki}^{-1}(\lambda_{li}(\tilde{y}_l))$ (well defined by construction of $\widetilde{W}_i$); then we get that:

$$\pi_j(\lambda_{kj}(\tilde{y}_k)) = \pi_k(\tilde{y}_k) = \pi_i(\lambda_{li}(\tilde{y}_l)) = \pi_l(\tilde{y}_l) = \pi_j(\lambda_{lj}(\tilde{y}_l)).$$

So by definition of uniformizing system there exists $g_1 \in G_j$ such that:

$$g_1 \circ \lambda_{kj}(\tilde{y}_k) = \lambda_{lj}(\tilde{y}_l);$$

then if we define $Q := [\lambda_{ki}, \tilde{y}_k, g_1 \circ \lambda_{kj}]$, we get that $s(P) = s(Q)$ and $t(P) = t(Q)$, so we can apply Lemma 3.12 and we get that there exists $g_2 \in G_j$ such that $P = [\lambda_{ki}, \tilde{y}_k, g_2 \circ g_1 \circ \lambda_{kj}]$. So we conclude that every point in $(s,t)^{-1}(\widetilde{W}_i \times \widetilde{W}_j)$ is of the form $[\lambda_{ki}, \tilde{y}_k, g \circ \lambda_{kj}]$ for some $\tilde{y}_k \in \widetilde{U}_k$ and some $g \in G_j$. So we have proved that:

$$(s,t)^{-1}(\widetilde{W}_i \times \widetilde{W}_j) \subseteq \bigcup_{g \in G_j} \widetilde{W}_k^{\lambda_{ki}, g \circ \lambda_{kj}} \tag{3.11}$$

(where we use the notation introduced in the proof of Proposition 3.10).

Now let us fix any compact set $K \subseteq (U \times U)$ and let us fix any sequence $\{P_n\}_{n \in \mathbb{N}}$ in $(s,t)^{-1}(K)$. If necessary by extracting a subsequence, we can assume that $(s,t)(P_n) =: (u_n, u'_n)$ converges to a point $(u, u') \in K$. Hence for every open neighborhood A of (u, u') in $U \times U$ we get that $(s,t)^{-1}(A)$ is not empty, so we are necessarily in the second of the previous 2 cases, hence there exists an open neighborhood $\widetilde{W}_i \times \widetilde{W}_j$ of (u, u') such that (3.11) holds. Now (u_n, u'_n) converges to (u, u'), so for n big enough we can assume that $P_n \in (s,t)^{-1}(\widetilde{W}_i \times \widetilde{W}_j)$; moreover, the union of (3.11) is made over a finite set, so by passing to a subsequence we can assume that there exists $g \in G_j$ such that $P_n \in \widetilde{W}_k^{\lambda_{ki}, g \circ \lambda_{kj}}$ for all n big enough. We have proved in Lemma 3.11 that s is a biholomorphism (hence homeomorphism) if restricted to this set, so if we call P the unique point of this set such that $s(P) = u$, we get that P_n converges to P. Hence we have proved that $(s,t)^{-1}$ of every compact set is compact, i.e. (s,t) is proper. □

From all the previous lemmas we get:

Proposition 3.15. *$R \overset{s}{\underset{t}{\rightrightarrows}} U$ is an object of* (**Grp**).

3.2. Morphisms and 2-morphisms

Our aim now is to associate to every compatible system (i.e. a morphism in (**Pre-Orb**)) a morphism in (**Grp**).

Definition 3.16. Let $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i \in I}$ and $\mathcal{V} = \{(\widetilde{V}_j, H_j, \phi_j)\}_{j \in J}$ be orbifold atlases for X and Y respectively, let $\tilde{f} : \mathcal{U} \to \mathcal{V}$ be a compatible system (see Definition 1.21) for a continuous map $f : X \to Y$ and let $R \overset{s}{\underset{t}{\rightrightarrows}} U$ and $R' \overset{s'}{\underset{t'}{\rightrightarrows}} U'$ be the groupoid objects associated to $\mathcal{U}$ and $\mathcal{V}$ respectively. For simplicity, from now on for every point $\tilde{x}_i \in \widetilde{U}_i$ we will denote with $\tilde{y}_i$ its image in $\widetilde{V}_i$ via the holomorphic function $\tilde{f}_{\widetilde{U}_i, \widetilde{V}_i} : \widetilde{U}_i \to \widetilde{V}_i$. Now we define a set map $\psi : U \to U'$ as:

$$\psi|_{\widetilde{U}_i} := \tilde{f}_{\widetilde{U}_i, \widetilde{V}_i} : \widetilde{U}_i \to \widetilde{V}_i \subseteq U'.$$

We define also a set map $\Psi : R \to R'$ as follows: for every point $[\lambda_{ki}, \tilde{x}_k, \lambda_{kj}] \in R$ and for every representative $(\lambda_{ki}, \tilde{x}_k, \lambda_{kj})$ of it we set:

$$\Psi([\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]) := [\tilde{f}(\lambda_{ki}), \tilde{f}_{\widetilde{U}_k, \widetilde{V}_k}(\tilde{x}_k), \tilde{f}(\lambda_{kj})] = [\tilde{f}(\lambda_{ki}), \tilde{y}_k, \tilde{f}(\lambda_{kj})].$$

Using Remark 3.6 and the fact that $\tilde{f}$ is a functor (by Definition 1.21), we get that Ψ does not depend on the representative chosen for $[\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]$.

Proposition 3.17. *(ψ, Ψ) is a morphism of groupoid objects in* (**Grp**) *from $R \overset{s}{\underset{t}{\rightrightarrows}} U$ to $R' \overset{s'}{\underset{t'}{\rightrightarrows}} U'$.*

Proof. First of all, since the local liftings of f are all holomorphic, so is ψ. Now we recall that in Proposition 3.10 we described a manifold atlas for R where the charts are of the form $(\widetilde{W}_k^{ij}, \phi_k^{ij})$; analogously, we can use similar charts of the form $(\widetilde{Z}_k^{ij}, \xi_k^{ij})$ on R'. If we write Ψ in coordinates with respect to these charts, we get that Ψ locally coincides with $\tilde{f}_{\widetilde{U}_k, \widetilde{V}_k}$, which is holomorphic by Definition 1.21. Hence in order to prove the statement it suffices to prove the axioms of Definition 2.2, which are easy to verify working set-theoretically, so we omit the details. □

Now let us fix two atlases $\mathcal{U}$ and $\mathcal{V}$ for X and Y respectively, two compatible systems $\tilde{f}_1, \tilde{f}_2 : \mathcal{U} \to \mathcal{V}$ for a continuous function $f : X \to Y$ and a natural transformation $\delta : \tilde{f}_1 \Rightarrow \tilde{f}_2$ as in Definition 1.24. Let us call $R \underset{t}{\overset{s}{\rightrightarrows}} U$ and $R' \underset{t'}{\overset{s'}{\rightrightarrows}} U'$ the groupoid objects associated to the atlases $\mathcal{U}$ and $\mathcal{V}$ respectively; moreover, let us denote with (ψ, Ψ) and (ϕ, Φ) the morphisms of groupoid objects from $(R \underset{t}{\overset{s}{\rightrightarrows}} U)$ to $(R' \underset{t'}{\overset{s'}{\rightrightarrows}} U')$ associated to $\tilde{f}_1$ and $\tilde{f}_2$ respectively by Proposition 3.16; then we have the following proposition:

Proposition 3.18. *The* set map: $\alpha : U = \coprod_{(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}} \widetilde{U}_i \to R'$ *defined by*:

$$\alpha(\tilde{x}_i, \widetilde{U}_i) := \left[1_{\widetilde{V}_i^1}, (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}_i^1}(\tilde{x}_i), \delta_{\widetilde{U}_i}\right]$$

is a natural transformation from (ψ, Ψ) *to* (ϕ, Φ) *in* (**Grp**).

Proof. First of all, *we claim that α is holomorphic*: if we restrict α to every open set $\widetilde{U}_i$ in U (with the natural chart $(\widetilde{U}_i, id)$), we get that its range is contained in the open set $A := q'((\widetilde{V}_i^1)^{1_{\widetilde{V}_i^1}, \delta_{\widetilde{U}_i}})$ (where $q' : \hat{R}' \to R'$ is the quotient map); this set is biholomorphic to $\widetilde{V}_i^1$ (see the proof of Proposition 3.10). By composing with these biholomorphic changes of coordinates we get that α coincides on $\widetilde{U}_i$ with the holomorphic map $(\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}_i^1}$. Since the open sets of the form $\widetilde{U}_i$ cover all U, we have proved that α is holomorphic on all U, i.e. it is a morphism in (**Manifolds**).

So in order to prove that α is a natural transformation in (**Grp**) it suffices to verify the axioms of Definition 2.5, but both are just consequences of some diagram chasing, so we omit the details. □

3.3. The 2-functor F

Until now we have described how to associate:

(i) to every orbifold atlas $\mathcal{U}$ a groupoid object $R \underset{t}{\overset{s}{\rightrightarrows}} U$, which is an object in (**Grp**);
(ii) to every compatible system $\tilde{f}$ a morphism (ψ, Ψ) of groupoid objects, which is in particular a morphism in (**Grp**);
(iii) to every natural transformation δ between compatible systems a natural transformation α in (**Grp**).

A straightforward calculation proves that:

Proposition 3.19. *Whenever we fix a pair of objects $\mathcal{U}, \mathcal{V}$ in* (**Pre-Orb**) *with associated groupoid objects $R \underset{t}{\overset{s}{\rightrightarrows}} U$ and $R' \underset{t'}{\overset{s'}{\rightrightarrows}} U'$ respectively, we get a functor*:

$$F = F_{\mathcal{U}, \mathcal{V}} : (\textbf{Pre-Orb})(\mathcal{U}, \mathcal{V}) \to (\textbf{Grp})\left(\left(R \underset{t}{\overset{s}{\rightrightarrows}} U\right), \left(R' \underset{t'}{\overset{s'}{\rightrightarrows}} U'\right)\right)$$

defined by (ii) *on the level of objects and by* (iii) *on the level of morphisms.*

Theorem 3.20. *The previous data define a* 2*-functor F from* (**Pre-Orb**) *to* (**Grp**).

Proof. It suffices to verify axioms (a), (b) and (c) of Definition 1.2.

(a) Let us fix any pair of compatible systems $\tilde{f} : \mathcal{U} \to \mathcal{V}$ and $\tilde{g} : \mathcal{V} \to \mathcal{W}$. Then for simplicity, let us call:

$$F(\mathcal{U}) =: \left(R \underset{t}{\overset{s}{\rightrightarrows}} U\right), \qquad F(\mathcal{V}) =: \left(R' \underset{t'}{\overset{s'}{\rightrightarrows}} U'\right), \qquad F(\mathcal{W}) =: \left(R'' \underset{t''}{\overset{s''}{\rightrightarrows}} U''\right),$$

$$F(\tilde{f}) =: (\psi, \Psi), \qquad F(\tilde{g}) =: (\phi, \Phi), \qquad \tilde{g} \circ \tilde{f} =: \tilde{h} \quad \text{and} \quad F(\tilde{h}) =: (\theta, \Theta).$$

So we want to prove that $\theta \overset{?}{=} \phi \circ \psi$ and $\Theta \overset{?}{=} \Phi \circ \Psi$. The proof of the second equality is a direct application of the definition; once this is proved, we can use Remark 2.3 and we get that:

$$\theta = s'' \circ \Theta \circ e = \left(s'' \circ \Phi\right) \circ (\Psi \circ e) = \phi \circ s' \circ e' \circ \psi = \phi \circ 1_{U'} \circ \psi = \phi \circ \psi.$$

(b) Let us fix a diagram of compatible systems and natural transformations in (**Pre-Orb**) of the form:

$$\begin{tikzcd} \mathcal{U} \arrow[r, bend left, "\tilde{f}_1"] \arrow[r, bend right, "\tilde{f}_2"'] \arrow[r, phantom, "\Downarrow\delta"] & \mathcal{V} \arrow[r, bend left, "\tilde{g}_1"] \arrow[r, bend right, "\tilde{g}_2"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{W}. \end{tikzcd}$$

For simplicity, let us use the notation of (a) on the level of objects and let us call:

$$F(\tilde{f}_i) =: (\psi_i, \Psi_i), \qquad F(\tilde{g}_i) =: (\phi_i, \Phi_i) \quad \text{for } i = 1, 2,$$
$$F(\delta) =: \alpha : U \to R', \qquad F(\eta) := \beta : U' \to R'' \quad \text{and} \quad F(\eta * \delta) := \gamma : U \to R''.$$

By definition of $*$ in (**Pre-Orb**), for every $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}$ we have that $(\eta * \delta)_{\widetilde{U}_i} = \eta_{\widetilde{V}_i^2} \circ \tilde{g}_1(\delta_{\widetilde{U}_i}) : \widetilde{W}_i^{11} \to \widetilde{W}_i^{22}$ (where we use the notation of Proposition 1.28); so for every point $(\tilde{x}_i, \widetilde{U}_i) \in U$ we have:

$$\gamma(\tilde{x}_i, \widetilde{U}_i) = \big[1_{\widetilde{W}_i^{11}}, (\tilde{g}_1 \circ \tilde{f}_1)_{\widetilde{U}_i, \widetilde{W}_i^{11}}(\tilde{x}_i), \eta_{\widetilde{V}_i^2} \circ \tilde{g}_1(\delta_{\widetilde{U}_i})\big]. \tag{3.12}$$

Moreover,

$$\begin{aligned} \big(F(\eta) * F(\delta)\big)(\tilde{x}_i, \widetilde{U}_i) &= (\beta * \alpha)(\tilde{x}_i, \widetilde{U}_i) \\ &= m''\big(\Phi_1 \circ \alpha(\tilde{x}_i, \widetilde{U}_i), \beta \circ \psi_2(\tilde{x}_i, \widetilde{U}_i)\big) \\ &= m''\big(\Phi_1\big(\big[1_{\widetilde{V}_i^1}, (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}_i^1}(\tilde{x}_i), \delta_{\widetilde{U}_i}\big]\big), \beta\big((\tilde{f}_2)_{\widetilde{U}_i, \widetilde{V}_i^2}(\tilde{x}_i), \widetilde{V}_i^2\big)\big) \\ &= m''\big(\big[1_{\widetilde{W}_i^{11}}, (\tilde{g}_1)_{\widetilde{V}_i^1, \widetilde{W}_i^{11}} \circ (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}_i^1}(\tilde{x}_i), \tilde{g}_1(\delta_{\widetilde{U}_i})\big], \\ &\quad \big[1_{\widetilde{W}_i^{21}}, (\tilde{g}_1)_{\widetilde{V}_i^2, \widetilde{W}_i^{21}} \circ (\tilde{f}_2)_{\widetilde{U}_i, \widetilde{V}_i^2}(\tilde{x}_i), \eta_{\widetilde{V}_i^2}\big]\big) \\ &= \big[1_{\widetilde{W}_i^{11}}, (\tilde{g}_1)_{\widetilde{V}_i^1, \widetilde{W}_i^{11}} \circ (\tilde{f}_1)_{\widetilde{U}_i, \widetilde{V}_i^1}(\tilde{x}_i), \eta_{\widetilde{V}_i^2} \circ \tilde{g}_1(\delta_{\widetilde{U}_i})\big]. \end{aligned} \tag{3.13}$$

By comparing (3.12) with (3.13), we get that $F(\eta) * F(\delta) = F(\eta * \delta)$.

The proof of (c) is straightforward, so we omit it. $\square$

4. *F* induces a bijection between equivalence classes of objects

4.1. Morita equivalences

We recall the following definition:

Definition 4.1. ([11], §2.4) A morphism $(\psi, \Psi) : (R' \overset{s'}{\underset{t'}{\rightrightarrows}} U') \to (R \overset{s}{\underset{t}{\rightrightarrows}} U)$ between Lie groupoids is called *Morita equivalence* (or *essential equivalence*) if the following 2 conditions hold:

(i) let us consider the following fiber product in (**Manifolds**):

$$\begin{tikzcd} R \times_U U' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\square"] & U' \arrow[d, "\psi"] \\ R \arrow[r, "s"'] & U \end{tikzcd} \tag{4.1}$$

since $R \overset{s}{\underset{t}{\rightrightarrows}} U$ is a Lie groupoid, we get that the map s is a submersion, so we can apply Proposition 2.10 and we get that the fiber product has a natural structure of complex manifold and that also π_2 is a submersion. *Then we require that the set map $t \circ \pi_1 : R \times_U U' \to U$ is a surjective holomorphic submersion*. This request makes sense because both source and target of this map are complex manifolds;

(ii) *we require also that the square*:

$$\begin{tikzcd} R' \arrow[r, "\Psi"] \arrow[d, "(s',t')"'] & R \arrow[d, "(s,t)"] \\ U' \times U' \arrow[r, "(\psi \times \psi)"'] & U \times U \end{tikzcd} \tag{4.2}$$

is cartesian in (**Manifolds**). Note that the square is always commutative because of Definition 2.2, so we have only to check the universal property of fiber products.

Definition 4.2. Two groupoid objects $R^i \rightrightarrows U^i$ (for $i = 1, 2$) in (**Manifolds**) are called *Morita equivalent* (or *weakly equivalent*) if there exist a third groupoid object $R^3 \rightrightarrows U^3$ and two Morita equivalences:

$$\left(R^1 \rightrightarrows U^1\right) \xleftarrow{(\psi,\Psi)} \left(R^3 \rightrightarrows U^3\right) \xrightarrow{(\phi,\Phi)} \left(R^2 \rightrightarrows U^2\right).$$

This is actually an equivalence relation, see for example [12], Chapter 5 for the proof.

Now let us fix any orbifold structure $\mathcal{X}$ on a topological space X and let us denote with $\mathcal{U} = \{(\widetilde{U}_i, G_i, \pi_i)\}_{i\in I}$ the maximal atlas associated to it by Definition 1.36. Then let us fix any other atlas $\mathcal{U}' = \{(\widetilde{U}_{i'}, G_{i'}, \pi_{i'})\}_{i'\in I'}$ in the class $\mathcal{X}$. By construction of $\mathcal{U}$, we get that $I' \subseteq I$, so by Remark 1.16 we get that $\mathcal{U}'$ is a subcategory of $\mathcal{U}$, hence we can consider a compatible system $\widetilde{id}: \mathcal{U}' \to \mathcal{U}$ over the identity of X as follows:

- as a functor, $\widetilde{id}$ is the inclusion on the level of objects and morphisms (i.e. uniformizing systems and embeddings);
- for every uniformizing system $(\widetilde{U}_{i'}, G_{i'}, \pi_{i'}) \in \mathcal{U}$ we set $\widetilde{id}_{\widetilde{U}_{i'},\widetilde{U}_{i'}} := 1_{\widetilde{U}_{i'}}$.

It is easy to see that all the axioms of Definition 1.21 are satisfied; as a useful notation, we will write an index as i' if it belongs to the set I' (and so also to I) and with i if it belongs to I and we don't know whether $(\widetilde{U}_i, G_i, \pi_i)$ belongs to $\mathcal{U}'$ or not.

Lemma 4.3. *The morphism $F(\widetilde{id})$ is a Morita equivalence.*

Proof. In order to prove such result, we use the following notation:

- $R \overset{s}{\underset{t}{\rightrightarrows}} U := F(\mathcal{U})$ and $R' \overset{s'}{\underset{t'}{\rightrightarrows}} U' := F(\mathcal{U}')$;
- $(\psi, \Psi) : (R' \overset{s'}{\underset{t'}{\rightrightarrows}} U') \to (R \overset{s}{\underset{t}{\rightrightarrows}} U)$ is the morphism $F(\widetilde{id})$.

A direct calculation shows that ψ is just the inclusion of $U' = \coprod_{i'\in I'} \widetilde{U}_{i'}$ in $U = \coprod_{i\in I} \widetilde{U}_i$ and that Ψ is the *holomorphic* map that associates to every point $[\lambda_{k'i'}, \tilde{x}_{k'}, \lambda_{k'j'}]$ the same point, but considered in R instead of R'. Moreover, set-theoretically (and up to natural bijection), we have that:

$$\begin{aligned} R \times_U U' &= \left\{\left(r, u'\right) \in R \times U' \text{ s.t. } s(r) = \psi\left(u'\right)\right\} \\ &= \left\{\left([\lambda_{ki'}, \tilde{x}_k, \lambda_{kj}], (\tilde{x}_{i'}, \widetilde{U}_{i'})\right) \text{ s.t. } \lambda_{ki'}(\tilde{x}_k) = \tilde{x}_{i'}\right\}. \end{aligned} \tag{4.3}$$

Now let us verify the 2 axioms of Definition 4.1.

(i) Let us consider the map $t \circ \pi_1$ defined on the fiber product (4.3). Then for every point in that set, we have:

$$t \circ \pi_1\left([\lambda_{ki'}, \tilde{x}_k, \lambda_{kj}], (\tilde{x}_{i'}, \widetilde{U}_{i'})\right) = t\left([\lambda_{ki'}, \tilde{x}_k, \lambda_{kj}]\right) = \left(\lambda_{kj}(\tilde{x}_k), \widetilde{U}_j\right).$$

We claim that $t \circ \pi_1$ *is surjective*, so let us fix any point $(\tilde{x}_j, \widetilde{U}_j) \in U$ and let us consider the point $\pi_j(\tilde{x}_j) \in X$; by hypothesis $\mathcal{U}'$ is an orbifold atlas for X, so there exists a uniformizing system $(\widetilde{U}_{i'}, G_{i'}, \pi_{i'})$ in $\mathcal{U}'$ for an open neighborhood of this point in X. Now $\mathcal{U}$ contains both $(\widetilde{U}_{i'}, G_{i'}, \pi_{i'})$ and $(\widetilde{U}_j, G_j, \pi_j)$, so by Remark 1.17 there exist a uniformizing system $(\widetilde{U}_k, G_k, \pi_k)$ in $\mathcal{U}$, a point $\tilde{x}_k$ in $\widetilde{U}_k$ and embeddings $\lambda_{ki'}$ and λ_{kj} such that $\lambda_{kj}(\tilde{x}_k) = \tilde{x}_j$. Then if we call $\tilde{x}_{i'} := \lambda_{ki'}(\tilde{x}_k)$ we get that the point $([\lambda_{ki'}, \tilde{x}_k, \lambda_{kj}], (\tilde{x}_{i'}, \widetilde{U}_{i'}))$ belongs to the fiber product (4.3) and its image via $t \circ \pi_1$ is $(\tilde{x}_j, \widetilde{U}_j)$, so our claim is proved.

Now by construction ψ is an embedding between manifolds of the same dimension, so in particular it is étale; we recall that (4.1) is a cartesian diagram, hence by Proposition 2.10 we get that also π_1 is étale. Moreover, t is étale by Lemma 3.11, hence $\pi_1 \circ t$ is étale, hence in particular it is a submersion. So we have completely proved condition (i).

(ii) Let us consider the following diagram:

$$\begin{tikzcd}
R' \arrow[dr, dashed, "\eta"] \arrow[drr, bend left, "\Psi"] \arrow[ddr, bend right, "(s',t')"'] & & \\
& R_{(s,t)} \times_{(\psi\times\psi)} U' \times U' \arrow[r, "pr_1"'] \arrow[d, "pr_2"] \arrow[dr, phantom, "\square"] & R \arrow[d, "(s,t)"] \\
& U' \times U' \arrow[r, "(\psi\times\psi)"'] & U \times U
\end{tikzcd} \tag{4.4}$$

where the external diagram is commutative because it is just diagram (4.2). Since both Ψ and (s',t') are holomorphic, by the universal property of the fiber product there exists a unique *holomorphic* map η that makes the diagram commute. It is easy to give an explicit description of the fiber product in (4.4) as the set:

$$\left\{\left([\lambda_{ki'}, \tilde{x}_k, \lambda_{kj'}], (\tilde{x}_{i'}, \widetilde{U}_{i'}), (\tilde{x}_{j'}, \widetilde{U}_{j'})\right) \text{ s.t. } \lambda_{ki'}(\tilde{x}_k) = \tilde{x}_{i'} \text{ and } \lambda_{kj'}(\tilde{x}_k) = \tilde{x}_{j'}\right\} \tag{4.5}$$

and of the set map η as:

$$[\lambda_{k'i'}, \tilde{x}_{k'}, \lambda_{k'j'}] \mapsto \left([\lambda_{k'i'}, \tilde{x}_{k'}, \lambda_{k'j'}], (\tilde{x}_{i'}, \widetilde{U}_{i'}), (\tilde{x}_{j'}, \widetilde{U}_{j'})\right).$$

Now by using Lemma 3.12 in R one can prove that η is surjective; the same lemma applied in R' proves that η is also injective. Since it is holomorphic, then it is a biholomorphism, therefore the external square of (4.4) is also a cartesian diagram, so (ii) is proved. $\square$

Proposition 4.4. *Suppose we have fixed two* equivalent *orbifold atlases $\mathcal{U}_1$ and $\mathcal{U}_2$ on a topological space X and let us call $R^i \overset{s^i}{\underset{t^i}{\rightrightarrows}} U^i$ (for $i = 1, 2$) the groupoid objects associated to them by the 2-functor F described in the previous chapter. Then these two groupoid objects are Morita equivalent.*

Proof. It suffices to consider the maximal manifold atlas $\mathcal{U}_3$ (associated to both) and to apply the previous lemma twice. $\square$

Proposition 4.5. *Suppose we have fixed two orbifold atlases $\mathcal{U}$ on X and $\mathcal{U}'$ on X' and assume that they are equivalent with respect to Definition* 1.37*. Then their images via the 2-functor F are Morita equivalent.*

Proof. The explicit construction of Section 3.1 proves that F identifies $\mathcal{U}$ and $\varphi^*(\mathcal{U})$ for every orbifold atlas $\mathcal{U}$ on X and any homeomorphism $\varphi: X' \to X$ (actually, this is the only point where F fails to be injective on objects). Then it suffices to apply the previous proposition on X' with equivalent orbifold atlases $\varphi^*(\mathcal{U})$ and $\mathcal{U}'$. $\square$

4.2. Surjectivity up to Morita equivalence

The aim of this subsection is to prove that the 2-functor F is surjective up to Morita equivalence. In order to do that, we divide the proof in some lemmas as follows.

Lemma 4.6. *Let us fix any proper étale and effective groupoid object $R \overset{s}{\underset{t}{\rightrightarrows}} U$ in* (**Manifolds**) *and let us define a relation of equivalence $\mathscr{R}$ on U as follows*:

$$\tilde{x}\mathscr{R}\tilde{y} \quad \overset{\text{def}}{\Longleftrightarrow} \quad \exists g \in R \quad \text{s.t.} \quad s(g) = \tilde{x} \quad \text{and} \quad t(g) = \tilde{y}.$$

Let us call $X := U/\mathscr{R}$ (with the quotient topology) and $\pi: U \to X$ the quotient map. Since (s,t) is proper, then X is Hausdorff and paracompact; we claim that we can define an orbifold atlas $\mathcal{U}$ on it.

Proof. (Adapted from [13], last part of Theorem 4.1.) Let us fix any point $\tilde{x} \in U$ and let us consider the set $R_{\tilde{x}} := (s,t)^{-1}\{(\tilde{x}, \tilde{x})\}$ in R. In Definition 2.15 we have already proved that this is a finite set and that for every point g in it we can find a sufficiently small open neighborhood W_g where s and t are both invertible; moreover, since $R_{\tilde{x}}$ is finite, we can restrict such neighborhoods such that $W_g \cap W_h = \varnothing$ for all $g \neq h$ in $R_{\tilde{x}}$. Let us define $\widetilde{A}_{\tilde{x}} := \bigcap_{g \in R_{\tilde{x}}} s(W_g)$, which is an open neighborhood of $\tilde{x}$ in U; now $R \smallsetminus \bigcup_{g \in R_{\tilde{x}}} W_g$ is closed so its image via (s,t) (which is proper, hence closed) is closed in $U \times U$ and does not contain $(\tilde{x}, \tilde{x})$. Since a basis for the topology of $U \times U$ is given by products of open sets of U, there exists $\widetilde{B}_{\tilde{x}}$ open neighborhood of $\tilde{x}$ contained in $\widetilde{A}_{\tilde{x}}$ and such that:

$$(\widetilde{B}_{\tilde{x}} \times \widetilde{B}_{\tilde{x}}) \cap (s,t)\left(R \smallsetminus \bigcup_{g \in R_{\tilde{x}}} W_g\right) = \varnothing. \tag{4.6}$$

Hence whenever $h \in R$ is such that both $s(h)$ and $t(h)$ belong to $\widetilde{B}_x$, then $h \in W_g$ for a *unique* $g \in R_{\tilde{x}}$. Now for every $g \in R_{\tilde{x}}$ we can define $\tilde{g} := t \circ (s|_{W_g})^{-1} : s(W_g) \to t(W_g)$, which is a biholomorphic map between open sets of U, both containing the point $\tilde{x}$. Since $\widetilde{B}_{\tilde{x}} \subseteq \widetilde{A}_{\tilde{x}}$, then $\widetilde{B}_{\tilde{x}} \subseteq s(W_g)$ for all $g \in R_{\tilde{x}}$, hence it makes sense to consider $\widetilde{C} := \bigcap_{g \in R_{\tilde{x}}} \tilde{g}(\widetilde{B}_{\tilde{x}})$ which is an open neighborhood of $\tilde{x}$ (since every $\tilde{g}$ is biholomorphic and maps $\tilde{x}$ to itself). Now let us fix any $h \in R_{\tilde{x}}$; if we apply part (a) of Lemma 2.16 we get that:

$$\tilde{h}(\widetilde{C}) = \tilde{h}\left(\bigcap_{g \in R_{\tilde{x}}} \tilde{g}(\widetilde{B}_{\tilde{x}})\right) = \bigcap_{g \in R_{\tilde{x}}} \tilde{h} \circ \tilde{g}(\widetilde{B}_{\tilde{x}}) = \bigcap_{m(g,h) \in R_{\tilde{x}}} \widetilde{m(g,h)}(\widetilde{B}_x) = \widetilde{C}.$$

Hence if we define:

$$G_{\tilde{x}} := \{\tilde{g}\colon \widetilde{C} \xrightarrow{\sim} \widetilde{C}\}_{g \in R_{\tilde{x}}},$$

this is a finite group (with multiplication given by composition of functions, or, equivalently, using Lemma 2.16) of holomorphic automorphisms on $\widetilde{C}$. If we call $\widetilde{U}_{\tilde{x}}$ the connected component of $\widetilde{C}$ that contains $\tilde{x}$, by continuity we get that $G_{\tilde{x}}$ is again a group of holomorphic automorphisms of $\widetilde{U}_{\tilde{x}}$. Moreover, using (4.6) one can easily prove that on $\widetilde{U}_{\tilde{x}}$ the relation $\mathscr{R}$ coincides with the equivalence relation induced by the action of $G_{\tilde{x}}$. Hence we get that $(\widetilde{U}_{\tilde{x}}, G_{\tilde{x}}, \pi)$ is a uniformizing system for the open set $\pi(\widetilde{U}_{\tilde{x}})$ of the topological space X.

Now we make the same construction also *for every open neighborhood* $\widetilde{B}'_{\tilde{x}} \subseteq \widetilde{B}_{\tilde{x}}$ *of* $\tilde{x}$ and we get uniformizing systems of the form $(\widetilde{U}'_{\tilde{x}}, G_{\tilde{x}}, \pi)$ with natural embeddings (given by inclusions) into the previous one. If we apply this construction for every point $\tilde{x} \in U$ we get a family $\mathcal{U}$ of *arbitrarily small* uniformizing systems for X which clearly satisfies axiom (i) of Definition 1.15. A direct check proves that this family satisfies also axiom (ii), so $\mathcal{U}$ is an orbifold atlas on X. □

For simplicity, we rename all the charts of $\mathcal{U}$ in order to make them of the form $(\widetilde{U}_i, G_i, \pi_i)$ (since we will not need to know the index "$\tilde{x}$" used to construct them). Let us call $R' \underset{t'}{\overset{s'}{\rightrightarrows}} U'$ the groupoid object associated to $\mathcal{U}$ by the 2-functor F and let us fix any point P in R'. Since $\mathcal{U}$ contains arbitrarily small open neighborhoods of every point, we get that P has the form $[\iota, \tilde{x}_k, \lambda_{kj}]$ where $\tilde{x}_k \in \widetilde{U}_k$ and $\iota : \widetilde{U}_k \to \widetilde{U}_i$ is just an inclusion. If we denote with $\tilde{x}_j := \lambda_{kj}(\tilde{x}_k)$, we have that $\pi_i(\tilde{x}_k) = \pi_j(\tilde{x}_j)$. By definition of the π_i's of the previous lemma, all these maps are simply restrictions of $\pi : U \twoheadrightarrow X := U/\mathscr{R}$. Therefore, there exists a point $h \in R$ such that $s(h) = \tilde{x}_k$ and $t(h) = \tilde{x}_j$. Eventually by restricting $\widetilde{U}_k$ we can assume that in U we have $\widetilde{U}_k \subseteq \tilde{h}^{-1}(\widetilde{U}_j)$, so it makes sense to consider $\tilde{h}$ as an open embedding from $\widetilde{U}_k$ to $\widetilde{U}_j$. Moreover, by applying the definition of this map it is easy to see that $\tilde{h}$ is an embedding from $(\widetilde{U}_k, G_k, \pi_k)$ to $(\widetilde{U}_j, G_j, \pi_j)$. Now let us consider the point $Q := [\iota, \tilde{x}_k, \tilde{h}]$; if we apply Lemma 3.12 (in R' instead of R) together with Lemma 2.16 we get the following result:

Lemma 4.7. *For every point* $P = [\lambda_{ki}, \tilde{x}_k, \lambda_{kj}]$ *of* R' *there exists a unique* g *in* R *such that*:

- $s(g) = \lambda_{ki}(\tilde{x}_k)$,
- $t(g) = \lambda_{kj}(\tilde{x}_k)$,
- $P = [\iota, \tilde{x}_k, \tilde{g}]$.

Our aim is to prove that there is a Morita equivalence (ψ, Ψ) from $R' \underset{t'}{\overset{s'}{\rightrightarrows}} U'$ to $R \underset{t}{\overset{s}{\rightrightarrows}} U$. First of all, let us define $\psi : U' \to U$ as $\psi(\tilde{x}_i, \widetilde{U}_i) := \tilde{x}_i$. Since U' is the disjoint union of open sets of the form $\widetilde{U}_i$, we get that in the natural coordinates this map locally coincides with the identity, so ψ is holomorphic.

Lemma 4.8. *Lemma* 4.7 *induces a biholomorphic map*:

$$\eta : R' \to R_{(s,t)} \times_{(\psi \times \psi)} U' \times U'.$$

Proof. Take any point P in R' and define $g \in R$ as in the previous lemma, so that $P = [\iota, \tilde{x}_k, \tilde{g}]$. Then define:

$$\eta\big([\iota, \tilde{x}_k, \tilde{g}]\big) := \big(g, s'(P), t'(P)\big).$$

Lemma 4.7 proves that η is well defined and that

$$(s,t)(g) = \big(\lambda_{ki}(\tilde{x}_k), \lambda_{kj}(\tilde{x}_k)\big) = (\psi \times \psi)\big(s'(P), t'(P)\big);$$

so we get that actually η has values in the fiber product. By the universal property of the fiber product, in order to prove that it is holomorphic it is sufficient to prove that η is so if composed with the natural projections pr_1 and pr_2 to R and $U' \times U'$ respectively. The second composition is just the map (s', t'), which is holomorphic, so let us consider only the first map, given by Lemma 4.7. It is obvious that for every $P = [\iota, \tilde{x}_k, \tilde{g}]$, the point g can be obtained as $(s_{|W_g})^{-1}(P)$; now for points P and $Q = [\iota', \tilde{y}_l, \tilde{h}]$ sufficiently near we get that h belongs to W_g, so h is equal to $(s_{|W_g})^{-1}(Q)$. Hence locally the map $pr_1 \circ \eta$ coincides with the local inverse of s, which is étale by hypothesis. So we have proved that η is holomorphic. In order to prove that it is biholomorphic, it suffices to prove that it is invertible. So let us describe explicitly its inverse:

$$\gamma : R_{(s,t)} \times_{(\psi \times \psi)} U' \times U' \to R'$$

as follows: let us take any point $(g, (\tilde{x}_i, \widetilde{U}_i)(\tilde{x}_j, \widetilde{U}_j))$ in the fiber product, so we have $s(g) = \psi(\tilde{x}_i, \widetilde{U}_i) = \tilde{x}_i$ and $t(g) = \tilde{x}_j$. Eventually by restricting the open neighborhood W_g of g in R, we can assume that $s(W_g) \subseteq \widetilde{U}_i$ and $t(W_g) \subseteq \widetilde{U}_j$. Hence it makes sense to consider $\tilde{g}$ as a map from an open neighborhood $\widetilde{U}_k$ of $\tilde{x}_i$ in $\widetilde{U}_i$ to $\widetilde{U}_j$. Moreover, this will be an embedding between uniformizing systems, so we can define:

$$\gamma\big(g, (\tilde{x}_i, \widetilde{U}_i), (\tilde{x}_j, \widetilde{U}_j)\big) := [\iota, \tilde{x}_i, \tilde{g}]$$

(where ι is the inclusion of $\widetilde{U}_k$ in $\widetilde{U}_i$); a direct check proves that γ does not depend on the choice of $\widetilde{U}_k$ and that it is the inverse of η. $\square$

Now we define the holomorphic map $\Psi := pr_1 \circ \eta : R' \to R$ and a direct computation shows that the pair (ψ, Ψ) is a morphism of groupoid objects from $R' \overset{s'}{\underset{t'}{\rightrightarrows}} U'$ to $R \overset{s}{\underset{t}{\rightrightarrows}} U$.

Lemma 4.9. *The morphism (ψ, Ψ) is a Morita equivalence.*

Proof. We have to verify axioms (i) and (ii) of Definition 4.1. First of all, let us consider the map $\psi : U' \to U$; we have already said that up to a holomorphic change of coordinates ψ locally coincides with the identity, hence ψ is étale. Moreover, it is clearly surjective (because the domains of the charts of $\mathcal{U}$ cover all U). Hence ψ is étale and surjective, so also the map π_1 of diagram (4.1) is étale (see Proposition 2.10) and surjective. Moreover, t is étale and surjective by definition of object in (**Grp**), hence $t \circ \pi_1$ is surjective and étale, so in particular it is a surjective submersion. Hence (i) is proved.

Now let us pass to (ii): from the previous constructions we get a commutative diagram of the form:

$$\begin{tikzcd}
R' \arrow[dr, "\eta"] \arrow[drr, bend left, "\Psi"] \arrow[ddr, bend right, "{(s',t')}"'] & & \\
& R_{(s,t)} \times_{(\psi\times\psi)} U' \times U' \arrow[r, "pr_1"'] \arrow[d, "pr_2"] \arrow[dr, phantom, "\square"] & R \arrow[d, "{(s,t)}"] \\
& U' \times U' \arrow[r, "(\psi\times\psi)"'] & U \times U.
\end{tikzcd}$$

The internal square is cartesian by construction; since η is a biholomorphism, we get that also the external diagram is cartesian, hence (ii) is proved. $\square$

Proposition 4.10. *The 2-functor F is surjective up to Morita equivalence.*

Proof. For every $R \overset{s}{\underset{t}{\rightrightarrows}} U$ in (**Grp**), we have described an orbifold atlas $\mathcal{U}$ and we have proved that $F(\mathcal{U}) = (R' \overset{s'}{\underset{t'}{\rightrightarrows}} U')$ is Morita equivalent to $R \overset{s}{\underset{t}{\rightrightarrows}} U$, so we are done. $\square$

4.3. The natural bijection on classes of objects in source and target

Using Proposition 4.5 one can induce from the 2-functor F a natural set map:

$$\tilde{F}:\left\{\begin{array}{l}\text{equivalence classes of}\\ \text{objects of } (\textbf{Pre-Orb})\\ \text{with respect to}\\ \text{Definition 1.37}\end{array}\right\}\to\left\{\begin{array}{l}\text{equivalence classes of}\\ \text{objects of } (\textbf{Grp})\\ \text{with respect to}\\ \text{Morita equivalences}\end{array}\right\}.$$

Our aim is to prove that the set map $\tilde{F}$ is a bijection. In order to do that, let us state and prove some preliminary results.

Lemma 4.11. *Let us fix any pair of orbifold atlases $\mathcal{U}=\{(\widetilde{U}_i,G_i,\pi_i)\}_{i\in I}$ on X and $\mathcal{U}'=\{(\widetilde{U}'_j,G'_j,\pi'_j)\}_{j\in J}$ on X', let us call $R\overset{s}{\underset{t}{\rightrightarrows}}U:=F(\mathcal{U})$ and $R'\overset{s'}{\underset{t'}{\rightrightarrows}}U':=F(\mathcal{U}')$ and let us suppose we have a Morita equivalence*:

$$(\psi,\Psi):\left(R'\overset{s'}{\underset{t'}{\rightrightarrows}}U'\right)\to\left(R\overset{s}{\underset{t}{\rightrightarrows}}U\right).$$

Then $\mathcal{U}$ and $\mathcal{U}'$ are equivalent with respect to Definition 1.37.

Proof. Let us recall that by definition of F, $U:=\coprod_{i\in I}\widetilde{U}_i$, so we can define a continuous map $\pi:U\to X$ that for every $i\in I$ coincides with the continuous map π_i on the connected component $\widetilde{U}_i$. Since π locally coincides with the maps π_i's, we get that π is not only continuous, but also open. Analogously, we can define a continuous open map $\pi':U'\to X'$. Now let us consider the diagram:

$$\begin{tikzcd} U' \arrow[r, "\psi"] \arrow[d, "\pi'"'] & U \arrow[d, "\pi"] \\ X' \arrow[r, dashed, "\varphi"'] & X; \end{tikzcd} \tag{4.7}$$

let us fix any point x' in X' and let $u',\bar{u}'\in U'$ be any pair of its preimages via π'. Then by definition of R', there exists a point r' in R' such that $s'(r')=u'$ and $t'(r')=\bar{u}'$. Then:

$$\psi(u')=\psi(s'(r'))=s(\Psi(r'))\quad\text{and}\quad\psi(\bar{u}')=\psi(t'(r'))=t(\Psi(r')).$$

So $\pi(\psi(u'))=\pi(\psi(\bar{u}'))$, hence we can induce a well-defined map $\varphi:X'\to X$ as $\varphi(\pi'(u')):=\pi(\psi(u'))$. Since ψ is continuous, so is φ by the universal property of the quotient topology. Now by [12], exercise 5.16.4, we get that ψ is étale, hence in particular it is open. Therefore, using diagram (4.7) we get that φ is also open.

Now we want to find an inverse δ for φ. In order to do that, let us fix any point $\pi(u)\in X$ and let us consider again diagram (4.1). Since $t\circ\pi_1$ is surjective, there exists a point $(r,u')\in R\times_U U'$ such that $t(r)=u$; the point we have chosen belongs to the fiber product, so $\psi(u')=s(r)$. Hence by definition of π we get that:

$$\pi(u)=\pi(t(r))=\pi(s(r))=\pi(\psi(u')).$$

So we have proved that every point in X is of the form $\pi(\psi(u'))$ for some $u'\in U'$; hence we define the set map δ as $\delta(\pi(\psi(u')))=\pi'(u')$. In order to prove that it is well defined, it is sufficient to use the fact that diagram (4.2) is cartesian by definition of Morita equivalence. A direct check proves that δ is actually the inverse of φ. In addition, it is continuous since φ is open, so φ is a homeomorphism. So we have proved condition (a) of Definition 1.37.

We want to prove also condition (b); in order to do that, it suffices to prove that for every point x' of X' there exist a pair of uniformizing systems in $\mathcal{U}'$ and $\varphi^*(\mathcal{U})$ that are compatible at this point. So let $x'=\pi'_j(\tilde{x}'_j,\widetilde{U}'_j)$ be any such point and let $(\tilde{x}_i,\widetilde{U}_i):=\Psi(\tilde{x}'_j,\widetilde{U}'_j)$, so that $\pi_i(\tilde{x}_i,\widetilde{U}_i)=\varphi(x')$. Now Ψ is continuous, therefore $\Psi(\widetilde{U}'_j)\subset\widetilde{U}_i$; moreover it is étale, so there exists a sufficiently small open neighborhood $\widetilde{A}$ of $\tilde{x}'_j$ in $\widetilde{U}'_j$ such that $\Psi|_{\widetilde{A}}$ is invertible. If we apply Lemma 1.9, we get a uniformizing system $(\widetilde{U}'',G'',\pi'')$ together with the inclusion ι of $\widetilde{U}''$ in $\widetilde{U}'_j$, such that π'' is just the restriction of π'_j. Then we get a holomorphic embedding $\widetilde{\Psi}:=\Psi\circ\iota$ from $\widetilde{U}''$ to $\widetilde{U}_i$; by definition of $\delta=\varphi^{-1}$ we have $\delta\circ\pi_i\circ\widetilde{\Psi}=\pi'_j|_{\widetilde{U}''}=\pi''$, therefore $\widetilde{\Psi}$ is an embedding of $(\widetilde{U}'',G'',\pi'')$ in $(\widetilde{U}_i,G_i,\varphi^{-1}\circ\pi_i)$. The first chart has also an embedding ι in $(\widetilde{U}'_j,G'_j,\pi'_j)$, so $\varphi^*(\mathcal{U})$ and $\mathcal{U}'$ are equivalent at x'. $\square$

Proposition 4.12. *Let us fix any pair of orbifold atlases $\mathcal{U}$ and $\mathcal{U}''$ on X and X'' respectively and let us suppose that $F(\mathcal{U})$ and $F(\mathcal{U}'')$ are Morita equivalent. Then $\mathcal{U}$ and $\mathcal{U}''$ are equivalent with respect to Definition* 1.37.

Proof. By Definition 4.2 we get that there exists an object $R \overset{s}{\underset{t}{\rightrightarrows}} U$ of (**Grp**) and a pair of Morita equivalences as follows:

$$F(\mathcal{U}) \leftarrow \left(R \overset{s}{\underset{t}{\rightrightarrows}} U\right) \rightarrow F\left(\mathcal{U}''\right).$$

Using the proof of Proposition 4.10 we get that there exist a space X, an orbifold atlas $\mathcal{U}'$ on it and a Morita equivalence:

$$F\left(\mathcal{U}'\right) \rightarrow \left(R \overset{s}{\underset{t}{\rightrightarrows}} U\right).$$

It is easy to prove that Morita equivalences are closed under composition, hence we get Morita equivalences:

$$F(\mathcal{U}) \leftarrow F\left(\mathcal{U}'\right) \rightarrow F\left(\mathcal{U}''\right).$$

Then it suffices to apply twice the previous lemma (and the fact that Definition 1.37 gives a relation of equivalence) in order to prove that $\mathcal{U}$ is equivalent to $\mathcal{U}''$. $\square$

Theorem 4.13. *The set map $\tilde{F}$ is a bijection.*

Proof. Surjectivity of the map $\tilde{F}$ is just Proposition 4.10; injectivity is Proposition 4.12. $\square$

Hence we have proved that the classes of complex reduced orbifold atlases (with respect to Definition 1.37) and the classes of proper étale effective groupoid objects (with respect to Morita equivalences) are different description of the same geometric objects. It will be very useful to prove that a similar result holds also on the level of morphisms and 2-morphisms, but for the moment we have no idea of how this can be made. If this can be done, we will have a very useful tool to translate results about orbifolds in results about groupoid objects, and conversely.

Appendix A. Some technical proofs

Here are the proofs of some technical lemmas cited in the previous sections.

Lemma 1.30. *The relation of Definition* 1.29 *is an equivalence relation.*

Proof. This relation is clearly symmetric and reflexive, so let us prove only transitivity. So let us suppose that we have 3 atlases $\mathcal{U}_1, \mathcal{U}_2$ and $\mathcal{U}_3$ on X with the second one equivalent to both the first and the third one. Let us fix any $x \in X$; by definition there exist:

- 2 uniformizing systems $(\widetilde{V}_1, H_1, \psi_1)$ and $(\widetilde{V}_3, H_3, \psi_3)$ around x;
- 2 uniformizing systems $(\widetilde{U}_i, G_i, \pi_i) \in \mathcal{U}_i$ for $i = 1, 3$ and 2 uniformizing systems $(\widetilde{U}_2, G_2, \pi_2), (\widetilde{U}'_2, G'_2, \pi'_2) \in \mathcal{U}_2$;
- 4 embeddings $\lambda_1, \lambda'_1, \lambda_3, \lambda'_3$ as follows:

$$\begin{tikzcd}
(\widetilde{U}_1, G_1, \pi_1) & & (\widetilde{U}_2, G_2, \pi_2) & & (\widetilde{U}'_2, G'_2, \pi) & & (\widetilde{U}_3, G_3, \pi_3). \\
& (\widetilde{V}_1, H_1, \psi_1) \arrow[ul, "\lambda_1"] \arrow[ur, "\lambda'_1"'] & & (\widetilde{V}_2, H_2, \psi_2) \arrow[ul, dashed, "\lambda_2"] \arrow[ur, dashed, "\lambda'_2"'] & & (\widetilde{V}_3, H_3, \psi_3) \arrow[ul, "\lambda_3"] \arrow[ur, "\lambda'_3"'] &
\end{tikzcd}$$

Since both $(\widetilde{U}_2, G_2, \pi_2)$ and $(\widetilde{U}'_2, G'_2, \pi'_2)$ belong to the atlas $\mathcal{U}_2$, there exists a third uniformizing system $(\widetilde{V}_2, H_2, \psi_2)$ around x in $\mathcal{U}_2$, together with embeddings λ_2, λ'_2 as in the previous diagram. If we have fixed any point $\tilde{x}_2 \in \widetilde{V}_2$ such that $\psi_2(\tilde{x}_2) = x$, by composing with a suitable automorphism of $\widetilde{U}_2$ there is no loss of generality in assuming that $\lambda_2(\tilde{x}_2)$ belongs to the image of λ'_1; hence the set $(\lambda'_1)^{-1}(\lambda_2(\widetilde{V}_2))$ is an open neighborhood of a preimage of x in $\widetilde{V}_1$. So by applying Lemma 1.9 we get a uniformizing system $(\widetilde{W}_1, K_1, \xi_1)$ around x, together with an embedding μ_1 of it into $(\widetilde{V}_1, H_1, \psi_1)$. Then the set map $\mu'_1 := \lambda_2^{-1} \circ \lambda'_1 \circ \mu_1$ is a well-defined holomorphic embedding of $(\widetilde{W}_1, K_1, \xi_1)$ into $(\widetilde{V}_2, H_2, \psi_2)$. Analogously, we can complete also the right part of the previous diagram and we obtain something of this form:

$$\begin{tikzcd}[column sep=small]
(\widetilde U_1, G_1, \pi_1) & & (\widetilde U_2, G_2, \pi_2) & & \big(\widetilde U'_2, G'_2, \pi\big) & & (\widetilde U_3, G_3, \pi_3). \\
& (\widetilde V_1, H_1, \psi_1) \arrow[ul, "\lambda_1"'] \arrow[ur, "\lambda'_1"] & & (\widetilde V_2, H_2, \psi_2) \arrow[ul, "\lambda_2"'] \arrow[ur, "\lambda'_2"] & & (\widetilde V_3, H_3, \psi_3) \arrow[ul, "\lambda_3"'] \arrow[ur, "\lambda'_3"] & \\
& & (\widetilde W_1, K_1, \xi_1) \arrow[ul, dashed, "\mu_1"'] \arrow[ur, dashed, "\mu'_1"] & & (\widetilde W_3, K_3, \xi_3) \arrow[ul, dashed, "\mu_3"'] \arrow[ur, dashed, "\mu'_3"] & &
\end{tikzcd}$$

By applying the previous construction another time, we get a diagram like this:

$$\begin{tikzcd}[column sep=small]
(\widetilde U_1, G_1, \pi_1) & & (\widetilde U_2, G_2, \pi_2) & & \big(\widetilde U'_2, G'_2, \pi\big) & & (\widetilde U_3, G_3, \pi_3). \\
& (\widetilde V_1, H_1, \psi_1) \arrow[ul, "\lambda_1"'] \arrow[ur, "\lambda'_1"] & & (\widetilde V_2, H_2, \psi_2) \arrow[ul, "\lambda_2"'] \arrow[ur, "\lambda'_2"] & & (\widetilde V_3, H_3, \psi_3) \arrow[ul, "\lambda_3"'] \arrow[ur, "\lambda'_3"] & \\
& & (\widetilde W_1, K_1, \xi_1) \arrow[ul, "\mu_1"'] \arrow[ur, "\mu'_1"] & & (\widetilde W_3, K_3, \xi_3) \arrow[ul, "\mu_3"'] \arrow[ur, "\mu'_3"] & & \\
& & & (\widetilde W_2, K_2, \xi_2) \arrow[ul, dashed, "\mu_2"'] \arrow[ur, dashed, "\mu'_2"] & & &
\end{tikzcd}$$

By considering the embeddings $\lambda_1 \circ \mu_1 \circ \mu_2$ and $\lambda'_3 \circ \mu'_3 \circ \mu'_2$ we get that the atlases $\mathcal{U}_1$ and $\mathcal{U}_3$ are equivalent at x. Since this holds for every point of X, we are done. $\square$

Lemma 3.7. *The map m is well defined.*

Proof. In order to prove the statement, we have to solve 2 problems:

(i) First of all, let us fix representatives $(\lambda_{ih}, \tilde{x}_i, \lambda_{ij})$ and $(\lambda_{kj}, \tilde{x}_k, \lambda_{kl})$ for the 2 points we have to "multiply". Our previous description of the multiplication map requires to *choose* a uniformizing system $(\widetilde U_f, G_f, \pi_f)$, a point $\tilde{x}_f \in \widetilde U_f$ and embeddings $\lambda_{fi}, \lambda_{fk}$ making (3.7) commute. However, this construction uses Lemma 3.3, which gives only the existence of such data, but not the uniqueness, so we have to verify that our construction does not depend on different completions of (3.6).

(ii) We have to prove that the multiplication does not depend on the representatives chosen for $[\lambda_{ih}, \tilde{x}_i, \lambda_{ij}]$ and for $[\lambda_{kj}, \tilde{x}_k, \lambda_{kl}]$.

Let us solve these problems separately.

(i) Let us suppose we can "complete" a diagram (3.6) in two different ways:

$$\begin{tikzcd}
& & \tilde{x}_r & & \\
& & \cap & & \\
& & \widetilde U_r \arrow[dl, "\lambda_{ri}"'] \arrow[dr, "\lambda_{rk}"] \arrow[d, phantom, "\curvearrowright"] & & \\
\widetilde U_h & \widetilde U_i \arrow[l, "\lambda_{ih}"'] \arrow[r, "\lambda_{ij}"] & \widetilde U_j & \widetilde U_k \arrow[l, "\lambda_{kj}"'] \arrow[r, "\lambda_{kl}"] & \widetilde U_l. \\
& & \widetilde U_f \arrow[ul, "\lambda_{fi}"] \arrow[ur, "\lambda_{fk}"'] \arrow[u, phantom, "\curvearrowright"] & & \\
& & \cup & & \\
& & \tilde{x}_f & &
\end{tikzcd} \tag{A.1}$$

Now we have that $\lambda_{ri}(\tilde{x}_r) = \lambda_{fi}(\tilde{x}_f)$, so we can apply Lemma 3.3 and we get that there exist a uniformizing system $(\widetilde U_s, G_s, \pi_s)$, a point $\tilde{x}_s \in \widetilde U_s$ and a pair of embeddings $\lambda_{sr}, \lambda_{sf}$ which make the following diagrams commute:

$$\begin{tikzcd}
 & \widetilde{U}_s \arrow[dl, "\lambda_{sr}"'] \arrow[dr, "\lambda_{sf}"] \arrow[dd, phantom, "\curvearrowright"] & \\
\widetilde{U}_r \arrow[dr, "\lambda_{ri}"'] & & \widetilde{U}_f \arrow[dl, "\lambda_{fi}"] \\
 & \widetilde{U}_i &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \tilde{x}_s \arrow[dl, mapsto, "\lambda_{sr}"'] \arrow[dr, mapsto, "\lambda_{sf}"] \arrow[dd, phantom, "\curvearrowright"] & \\
\tilde{x}_r \arrow[dr, mapsto, "\lambda_{ri}"'] & & \tilde{x}_f . \arrow[dl, mapsto, "\lambda_{fi}"] \\
 & \tilde{x}_i &
\end{tikzcd} \tag{A.2}$$

Now using (A.1) and (A.2) together we get that $\lambda_{kj} \circ \lambda_{fk} \circ \lambda_{sf} = \lambda_{kj} \circ \lambda_{rk} \circ \lambda_{sr}$ and we recall that λ_{kj} is an embedding, hence in particular it is injective, so we have that:

$$\lambda_{fk} \circ \lambda_{sf} = \lambda_{rk} \circ \lambda_{sr}. \tag{A.3}$$

Now if we combine together diagram (A.2) and Eq. (A.3), we get commutative diagrams:

$$\begin{tikzcd}
 & \widetilde{U}_r \arrow[dl, "\lambda_{ih}\circ\lambda_{ri}"'] \arrow[dr, "\lambda_{kl}\circ\lambda_{rk}"] & \\
\widetilde{U}_h \arrow[r, phantom, "\curvearrowright"] & \widetilde{U}_s \arrow[u, "\lambda_{sr}"'] \arrow[d, "\lambda_{sf}"] \arrow[r, phantom, "\curvearrowright"] & \widetilde{U}_l \\
 & \widetilde{U}_f \arrow[ul, "\lambda_{ih}\circ\lambda_{fi}"] \arrow[ur, "\lambda_{kl}\circ\lambda_{fk}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \tilde{x}_r \arrow[dl, mapsto, "\lambda_{ih}\circ\lambda_{ri}"'] \arrow[dr, mapsto, "\lambda_{kl}\circ\lambda_{rk}"] & \\
\tilde{x}_h := \lambda_{hi}(\tilde{x}_i) \arrow[r, phantom, "\curvearrowright"] & \tilde{x}_s \arrow[u, mapsto, "\lambda_{sr}"'] \arrow[d, mapsto, "\lambda_{sf}"] \arrow[r, phantom, "\curvearrowright"] & \tilde{x}_l := \lambda_{kl}(\tilde{x}_k). \\
 & \tilde{x}_f \arrow[ul, mapsto, "\lambda_{ih}\circ\lambda_{fi}"] \arrow[ur, mapsto, "\lambda_{kl}\circ\lambda_{fk}"'] &
\end{tikzcd} \tag{A.4}$$

This means that $(\lambda_{ih} \circ \lambda_{fi}, \tilde{x}_f, \lambda_{kl} \circ \lambda_{fk}) \sim (\lambda_{ih} \circ \lambda_{ri}, \tilde{x}_r, \lambda_{kl} \circ \lambda_{rk})$, hence (i) is solved.

(ii) Let us suppose we have chosen another representative the point $(\lambda_{sh}, \tilde{x}_s, \lambda_{sj})$ for $[\lambda_{ih}, \tilde{x}_i, \lambda_{ij}]$. Using Remark 3.6 it suffices to consider the case when the two representatives are related by a diagram of the form (3.5); in other words, we can assume there exists an embedding λ_{si} such that:

$$\lambda_{sh} = \lambda_{ih} \circ \lambda_{si}, \qquad \lambda_{sj} = \lambda_{ij} \circ \lambda_{si} \quad \text{and} \quad \lambda_{si}(\tilde{x}_s) = \tilde{x}_i.$$

Now if we want to compute $m([\lambda_{ih}, \tilde{x}_i, \lambda_{ij}], [\lambda_{kj}, \tilde{x}_k, \lambda_{kl}])$ using this new representative for the first point, we have to use Lemma 3.3 in order to choose a uniformizing system $(\widetilde{U}_r, G_r, \pi_r)$ together with a point $\tilde{x}_r \in \widetilde{U}_r$ and a pair of embeddings $\lambda_{rs}, \lambda_{rk}$ such that:

$$\lambda_{sj} \circ \lambda_{rs} = \lambda_{kj} \circ \lambda_{rk}, \qquad \lambda_{rs}(\tilde{x}_r) = \tilde{x}_s \quad \text{and} \quad \lambda_{rk}(\tilde{x}_r) = \tilde{x}_k.$$

Note that there are no problems in choosing all these data, since we have already proved (i). In other words, we are using commutative diagrams of the form:

$$\begin{tikzcd}
 & \widetilde{U}_s \arrow[dl, "\lambda_{sh}"'] \arrow[d, "\lambda_{si}"] \arrow[dr, "\lambda_{sj}"] \arrow[dl, phantom, shift left=4, "\curvearrowright"] \arrow[dr, phantom, shift right=4, "\curvearrowright"] & \widetilde{U}_r \arrow[l, "\lambda_{rs}"'] \arrow[dr, "\lambda_{rk}"] \arrow[d, phantom, "\curvearrowright"] & & \\
\widetilde{U}_h & \widetilde{U}_i \arrow[l, "\lambda_{ih}"] \arrow[r, "\lambda_{ij}"] & \widetilde{U}_j \arrow[d, phantom, "\curvearrowright"] & \widetilde{U}_k \arrow[l, "\lambda_{kj}"'] \arrow[r, "\lambda_{kl}"] & \widetilde{U}_l \\
 & & \widetilde{U}_f \arrow[ul, "\lambda_{fi}"] \arrow[ur, "\lambda_{fk}"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \tilde{x}_s \arrow[dl, "\lambda_{sh}"'] \arrow[d, "\lambda_{si}"] \arrow[dr, "\lambda_{sj}"] & \tilde{x}_r \arrow[l, "\lambda_{rs}"'] \arrow[dr, "\lambda_{rk}"] & & \\
\tilde{x}_h & \tilde{x}_i \arrow[l, "\lambda_{ih}"'] \arrow[r, "\lambda_{ij}"] & \tilde{x}_j & \tilde{x}_k \arrow[l, "\lambda_{kj}"'] \arrow[r, "\lambda_{kl}"] & \tilde{x}_l \\
 & & \tilde{x}_f \arrow[ul, "\lambda_{fi}"] \arrow[ur, "\lambda_{fk}"'] & &
\end{tikzcd}$$

where for simplicity we have used the following notation: $\tilde{x}_h := \lambda_{sh}(\tilde{x}_s) = \lambda_{ih} \circ \lambda_{si}(\tilde{x}_s)$, $\tilde{x}_l := \lambda_{kl}(\tilde{x}_k)$ and $\tilde{x}_j := \lambda_{ij}(\tilde{x}_i)$. So we get the diagram:

$$\begin{tikzcd}
 & & \tilde{x}_r \arrow[d, phantom, "\Cap"] & & \\
 & & \widetilde{U}_r \arrow[dl, "\lambda_{ri} := \lambda_{si} \circ \lambda_{rs}"'] \arrow[dr, "\lambda_{rk}"] & & \\
\widetilde{U}_h & \widetilde{U}_i \arrow[l, "\lambda_{ih}"'] \arrow[r, "\lambda_{ij}"] & \widetilde{U}_j & \widetilde{U}_k \arrow[l, "\lambda_{kj}"'] \arrow[r, "\lambda_{kl}"] & \widetilde{U}_l \\
 & & \widetilde{U}_f \arrow[ul, "\lambda_{fi}"] \arrow[ur, "\lambda_{fk}"'] & & \\
 & & \tilde{x}_f \arrow[u, phantom, "\Cup"] & &
\end{tikzcd}$$

with $\lambda_{ri}(\tilde{x}_r) = \lambda_{fi}(\tilde{x}_f)$, so we can repeat the same construction of (i) in order to get a diagram of the form (A.4), so we obtain:

$$(\lambda_{ih} \circ \lambda_{ri}, \tilde{x}_r, \lambda_{kl} \circ \lambda_{rk}) \sim (\lambda_{ih} \circ \lambda_{fi}, \tilde{x}_f, \lambda_{kl} \circ \lambda_{fk}).$$

Now by definition of λ_{ri} we have $\lambda_{ih} \circ \lambda_{ri} = \lambda_{ih} \circ \lambda_{si} \circ \lambda_{rs} = \lambda_{sh} \circ \lambda_{rs}$, hence:

$$(\lambda_{sh} \circ \lambda_{rs}, \tilde{x}_r, \lambda_{kl} \circ \lambda_{rk}) \sim (\lambda_{ih} \circ \lambda_{fi}, \tilde{x}_f, \lambda_{kl} \circ \lambda_{fk}).$$

These two points are the multiplication obtained when we choose representatives $(\lambda_{ih}, \tilde{x}_i, \lambda_{ij})$ and $(\lambda_{sh}, \tilde{x}_s, \lambda_{sj})$ for the same point, so the multiplication does not depend on the representative chosen for $[\lambda_{ih}, \tilde{x}_i, \lambda_{ij}]$. In the same way one can also prove that the multiplication doesn't depend on the representative chosen for the point $[\lambda_{kj}, \tilde{x}_k, \lambda_{kl}]$. $\square$

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